The **production function** is a function that gives the maximum output attainable from a given combination of inputs. The production function is defined as

\[ f(x) = \max_y \left[ y: (x, y) \text{ is an element of the production set} \right] \]

\[ = \max_{y \in P(x)} [y] \]

where \( y \) represents output, \( x_j \) is the quantity used of the \( j \)th input, \( (x_1, x_2, x_3, \ldots, x_n) \) is the input bundle, \( n \) is the number of inputs used by the firm, and \( f(\cdot) \) represents the functional relationship between \( y \) and \( (x_1, x_2, x_3, \ldots, x_n) \).

**Marginal (physical) product** is defined as the increment in production that occurs when an additional unit of one particular input is employed. It is defined as

\[ MP_i = \frac{\Delta y}{\Delta x_i} = \frac{y^1 - y^0}{x_i^1 - x_i^0} \]

where \( y^1 \) and \( x^1 \) are the level of output and input after the change in the input level and \( y^0 \) and \( x^0 \) are the levels before the change in input use.

An **isoquant** curve in 2 dimensions represents all combinations of two inputs that produce the same quantity of output. The word “iso” means the same and “quant” stands for quantity of output. Isoquants are contour lines of the production function. Isoquants are analogous to indifference curves. Indifference curves represent combinations of goods that yield the same utility, isoquants represent combinations of inputs that yield the same level of production.

**Properties of isoquants**
- Isoquants slope down (have a negative slope).
- Higher (further from the origin) isoquants represent greater levels of output than lower isoquants.
- Isoquants are convex to the origin.
- The slope of an isoquant is called the marginal rate of (technical) substitution between input 1 and input 2 and tells us the decrease in the quantity of input 1 (\( x_1 \)) that is needed to accompany a one unit increase in the quantity of input two (\( x_2 \)), in order to keep production the same. We denote this rate of substitution as

\[ MRS_{x_1, x_2} \text{ or } MRTS_{x_1, x_2} \].

Algebraically we define this as

\[ MRS_{x_1, x_2} = \frac{\Delta x_1}{\Delta x_2} \cdot \text{y=constant} \]

**Slope of isoquants (MRS) and marginal physical products**

All points on an isoquant are associated with the same amount of production. Hence the loss in production associated with \( \Delta x_1 \) must equal the gain in production from \( \Delta x_2 \) as we increase the level of \( x_2 \) and decrease the level of \( x_1 \). Using algebra we can express this as

\[ MPP_{x_1} \Delta x_1 + MPP_{x_2} \Delta x_2 = 0 \]

We can rearrange this expression by subtracting \( MPP_{x_2} \Delta x_2 \) from both sides, and then dividing both sides first by \( MPP_{x_1} \) and then by \( x_2 \). This will give

\[ MPP_{x_1} \Delta x_1 + MPP_{x_2} \Delta x_2 = 0 \]

\[ \Rightarrow MPP_{x_1} \Delta x_1 = -MPP_{x_2} \Delta x_2 \]

\[ \Rightarrow \Delta x_1 = \frac{-MPP_{x_2} \Delta x_2}{MPP_{x_1}} \]

\[ \Rightarrow \frac{\Delta x_1}{\Delta x_2} = \frac{-MPP_{x_2}}{MPP_{x_1}} = MRS_{x_1, x_2} \]
The cost minimization problem

For each possible level of output, the firm will choose the one of several combinations of inputs that has the lowest cost. We can write this problem as

\[ C(y, w_1, w_2, \ldots) = \min_{x_1, x_2, \ldots, x_n} \sum_{i=1}^{n} w_i x_i \text{ such that } y = f(x_1, x_2, \ldots, x_n) \]

For each \( y \), the firm will choose different levels of each of the inputs.

Isocost lines

An isocost line identifies the combinations of inputs the firm can afford to buy with a given expenditure or cost (\( C \)), at given input prices. With \( n \) inputs the isocost line is given by

\[ w_1 x_1 + w_2 x_2 + \cdots + w_n x_n = C. \]

With two inputs the isocost line is given by \( w_1 x_1 + w_2 x_2 = C \). The slope of the isocost line is given by \( \frac{w_2}{w_1} \). We can obtain this by solving the isocost equation for \( x_1 \).

Statement of optimality conditions

The optimum point is on the isocost line

The optimum point is on isoquant

The isoquant and the isocost line are tangent at the optimum combination of \( x_1 \) and \( x_2 \).

The slope of the isocost line and the slope of the isoquant are equal at the optimum which implies that

\[ \frac{-w_2}{w_1} = MRS_{x_1, x_2} = \frac{- MPP_{x_2}}{MPP_{x_1}} \]

The ratio of marginal products is equal to the ratio of prices, i.e.,

\[ \frac{w_2}{w_1} = \frac{MPP_{x_2}}{MPP_{x_1}} \]

The marginal product of each input divided by its price is equal to the marginal product of every other input divided by its price, i.e.,

\[ \frac{MPP_{x_1}}{w_2} = \frac{MPP_{x_2}}{w_1}. \]

Example table \((y = 10,000)\)

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