Cost Minimization and Isoquants
Handout

The **production function** is a function that gives the maximum output attainable from a given combination of inputs. The production function is defined as

$$f(x) = \max_{y} \; [y\text{: } (x, y) \;\; \textit{is an element of the production set}]$$
$$= \max_{y \,\varepsilon\, P(x)} \; [y]$$

where y represents output, x_j is the quantity used of the jth input, $(x_1, x_2, x_3, \ldots x_n)$ is the input bundle, n is the number of inputs used by the firm, and f(·) represents the functional relationship between y and $(x_1, x_2, x_3, \ldots x_n)$.

Marginal (physical) product is defined as the increment in production that occurs when an additional unit of one particular input is employed. It is defined as $MP_i = \frac{\Delta y}{\Delta x_i} = \frac{y^1 - y^0}{x_i^1 - x_i^0}$ where y^1 and x^1 are the level of output and input after the change in the input level and y^0 and x^0 are the levels before the change in input use.

An **isoquant** curve in 2 dimensions represents all combinations of two inputs that produce the same quantity of output. The word "iso" means the same and "quant" stands for quantity of output. Isoquants are contour lines of the production function. Isoquants are analogous to indifference curves. Indifference curves represent combinations of goods that yield the same utility, isoquants represent combinations of inputs that yield the same level of production.

Properties of isoquants

Isoquants slope down (have a negative slope).
Higher (further from the origin) isoquants represent greater levels of output than lower isoquants
Isoquants are convex to the origin.
The slope of an isoquant is called the marginal rate of (technical) substitution between input 1 and input 2 and tells us the decrease in the quantity of input 1 (x_1) that is needed to accompany a one unit increase in the quantity of input two (x_2), in order to keep production the same. We denote this rate of substitution as MRS_{x_1, x_2} or MRTS_{x_1, x_2}. Algebraically we define this as $MRS_{x_1,x_2} = \frac{\Delta x_1}{\Delta x_2} \Big|_{y = constant}$.

Slope of isoquants (MRS) and marginal physical products

All points on an isoquant are associated with the same amount of production. Hence the loss in production associated with Δx_1 must equal the gain in production from Δx_2 as we increase the level of x_2 and decrease the level of x_1. Using algebra we can express this as $MPP_{x_1} \, \Delta x_1 + MPP_{x_2} \, \Delta x_2 = 0$. We can rearrange this expression by subtracting $MPP_{x_2} \, \Delta x_2$ from both sides, and then dividing both sides first by MPP_{x_1} and then by x_2. This will give

$$MPP_{x_1} \, \Delta x_1 + MPP_{x_2} \, \Delta x_2 = 0$$
$$\Rightarrow MPP_{x_1} \, \Delta x_1 = -MPP_{x_2} \, \Delta x_2$$
$$\Rightarrow \Delta x_1 = \frac{-MPP_{x_2} \, \Delta x_2}{MPP_{x_1}}$$
$$\Rightarrow \frac{\Delta x_1}{\Delta x_2} = \frac{-MPP_{x_2}}{MPP_{x_1}} = MRS_{x_1 x_2}$$

The cost minimization problem

For each possible level of output, the firm will choose the one of several combinations of inputs that has the lowest cost. We can write this problem as

$$C(y,\ w_1, w_2, \cdots) \ = \ \min_{x_1, x_2, \cdots, x_n} \ \sum_{i=1}^{n} w_i x_i \quad such\ that\ y \ = \ f(x_1, x_2, \cdots x_n)$$

For each y, the firm will choose different levels of each of the inputs.

Isocost lines

An **isocost line** identifies the combinations of inputs the firm can afford to buy with a given expenditure or cost (C), at given input prices. With n inputs the isocost line is given by $w_1 x_1 + w_2 x_2 + w_3 x_3 + \cdots + w_n x_n = C$. With two inputs the isocost line is given by $w_1x_1 + w_2x_2 = C$. The slope of the isocost line is given by $\frac{-w_2}{w_1}$. We can obtain this by solving the isocost equation for x_1.

Statement of optimality conditions

The optimum point is on the isocost line
The optimum point is on isoquant
The isoquant and the isocost line are tangent at the optimum combination of x_1 and x_2.
The slope of the isocost line and the slope of the isoquant are equal at the optimum which implies that

$$\frac{-w_2}{w_1} = MRS_{q_1 q_2} = \frac{-MPP_{x_2}}{MPP_{x_1}}$$

The ratio of marginal products is equal to the ratio of prices, i.e., $\frac{w_2}{w_1} = \frac{MPP_{x_2}}{MPP_{x_1}}$

The marginal product of each input divided by its price is equal to the marginal product of every other input divided by its price, i.e., $\frac{MPP_{x_2}}{w_2} = \frac{MPP_{x_1}}{w_1}$.

Example table (y = 10,000)

x_1	x_2	Approx MRS	MPP_1	MPP_2	MPP_1/w_1	MPP_2/w_2	MRS	$-w_2/w_1$
–	1.0000	–	0.0000	17.0000	0.0000	0.8500	→	-3.3333
–	2.0000	–	200.0000	48.0000	33.3333	2.4000	-0.2400	-3.3333
–	3.0000	–	400.0000	73.0000	66.6667	3.6500	-0.1825	-3.3333
12.4687	4.0000	–	664.6851	2585.7400	110.7809	129.2870	-3.8902	-3.3333
11.8528	**4.1713**	**-3.5946**	**739.5588**	**2465.2134**	**123.2598**	**123.2607**	**-3.3334**	**-3.3333**
9.7255	5.0000	-2.5672	1010.5290	2050.0940	168.4215	102.5047	-2.0287	-3.3333
9.3428	5.1972	-1.9411	1063.1321	1975.4051	177.1887	98.7703	-1.8581	-3.3333
8.0629	6.0000	-1.5941	1254.9695	1724.5840	209.1616	86.2292	-1.3742	-3.3333
6.9792	6.9063	-1.1959	1447.3307	1508.9951	241.2218	75.4498	-1.0426	-3.3333
6.8827	7.0000	-1.0291	1466.3867	1489.5380	244.3978	74.4769	-1.0158	-3.3333
5.9898	8.0000	-0.8929	1663.9176	1305.9560	277.3196	65.2978	-0.7849	-3.3333
5.2904	9.0000	-0.6994	1855.3017	1155.0760	309.2169	57.7538	-0.6226	-3.3333
4.7309	10.0000	-0.5595	2044.1823	1026.1700	340.6971	51.3085	-0.5020	-3.3333
4.2773	11.0000	-0.4535	2232.4134	912.4680	372.0689	45.6234	-0.4087	-3.3333
3.9071	12.0000	-0.3702	2420.9761	809.4240	403.4960	40.4712	-0.3343	-3.3333
3.6042	13.0000	-0.3029	2610.3937	713.8360	435.0656	35.6918	-0.2735	-3.3333