Linear Functions and their Slopes

Functions

A ***function*** of a real variable x (with **domain** D) is a **rule** that assigns a **unique** real number to each number x in the domain D. We sometimes use the word **mapping** instead of function. Functions are often given letter names such as f, g, F, or f . We often call x the independent variable or the argument of the function f. If g is the function and x is a number in D, then g(x) denotes the number that g assigns to x. We can also write $x \mapsto g(x)$ meaning that x is transformed into another number g(x) by the function g. We sometimes make the idea that g has an argument (we substitute a number for the variable in g) explicit by writing $g(\cdot)$. We often use a variable name to represent the value of the function $f(\cdot)$. For example we might write y = f(x) where x represents a number in the domain of f. Or we might write p = g(q) where q represents a number in the domain of g.

Examples of functions

1. Let f(x) = 3x + 2.

   First consider the case where x = 4.

$$\text{If} \quad x = 4 \quad \text{then} \quad f(x) = (3)(4) + 2 = 12 + 2 = 14.$$

   Now let x = -2.

$$\text{If} \quad x = -2 \quad \text{then} \quad f(x) = (3)(-2) + 2 = -6 + 2 = -4.$$

2. Let t = g(x) = $2x^2 + 3x - 4$.

   First consider the case where x = 4.

$$\text{If} \quad x = 4 \quad \text{then} \quad t = g(x) = (2)(4)^2 + (3)(4) - 4 = 32 + 12 - 4 = 40.$$

   Now let x = -3.

$$\text{If} \quad x = -3 \quad \text{then} \quad t = g(x) = (2)(-3)^2 + (3)(-3) - 4 = 18 - 9 - 4 = 5.$$

3. Let p = h(q) = $(2q^2)$ (3q) + 2q.

   First consider the case where q = 4.

$$\text{If} \quad q = 4 \quad \text{then} \quad p = h(q) = (2)(4)^2\,(3)(4) + (2)(4) = (32)\,(12) + 8 = 384 + 8 = 392.$$

   Now let q = -1.

$$\text{If} \quad q = -1 \quad \text{then} \quad p = h(x) = (2)(-1)^2\,(3)(-1) + (2)(-1)$$
$$= (2)\,(-3) + -2 = -6 - 2 = -8.$$

The domain of a function

The domain of a function is the set of all values that can be substituted for x in the function $f(\cdot)$. If a function f is defined using an algebraic formula, we normally adopt the convention that the domain consists of all values of the independent variable for which the function gives a meaningful value (unless the domain is explicitly mentioned).

Examples

1. $y = f(x) = 6x^2 + 2$

   Because $f(\cdot)$ is defined for all real numbers we say that the domain of f is the real line or all numbers between $-\infty$ and ∞.

2. $z = g(w) = w^2 + \frac{1}{w}$

   Because $g(\cdot)$ is not defined at w = 0, but is defined for all other real numbers, we say that the domain of g is all numbers between $-\infty$ and ∞ except for zero.

3. $p = f(q) = \sqrt{q} + 2q$

   Because $f(\cdot)$ is not defined if q is a negative number, but is defined for all non-negative real numbers we say that the domain of f is all non-negative real numbers.

   However, $f(\cdot)$ is not really a function because it does not assign a **unique** real number to each q. For q = 9, p can take the value 9 (3 + 6) or the value 3 (-3 + 6). If we let $p = f(q) = +\sqrt{q} + 2q$ then f is a function and the domain above is correct.

The range of a function

Let g be a function with domain D. The set of all values g(x) that the function assumes is called the range of g.

Examples

1. $y = f(x) = 6x^2 + 2, \quad x \text{ is any real number}$

   The range of the function will be all real numbers greater than or equal to 2. For example, if x = -3, then y = f(x) = 56. Or if x = 0.1 then y = f(x) = 2.06.

2. $z = g(w) = w^2 + \frac{1}{w}, \quad x \neq 0$

   The range of the function is all real numbers. For example if w = -3, then $z = g(w) = 9 + \frac{1}{-3} = 8\frac{2}{3}$.

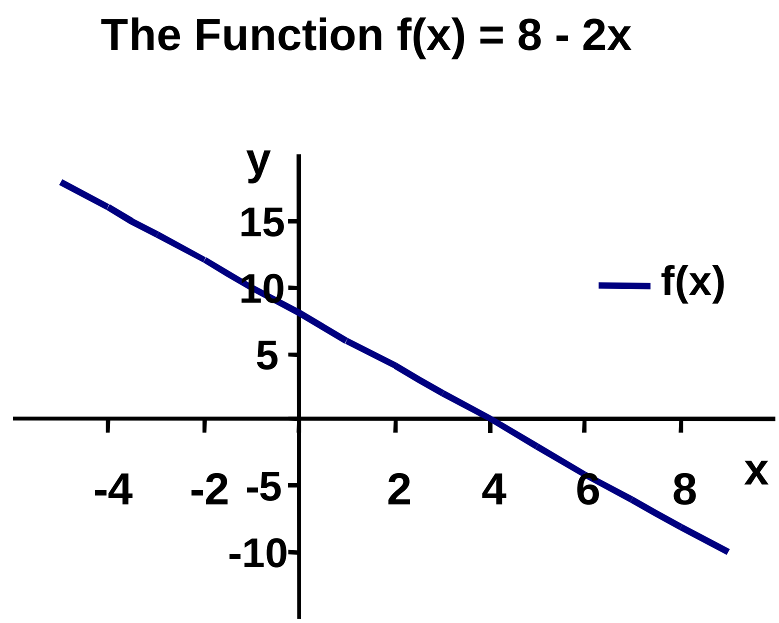
   Or if w = - 0.1 then $z = g(w) = 0.01 + \frac{1}{-0.1} = 0.01 + -10 = -9.99$.

The graph of a function in 2 dimensions

When the rule that defines a function g(·) is given by an equation in y and x, the graph of g is the graph of the equation, that is the set of points (x, y) in the xy-plane that satisfies the equation. Another way to say this is that the graph of the function g(·) is the set of all point [x, g(x)], where x belongs to the domain of g.

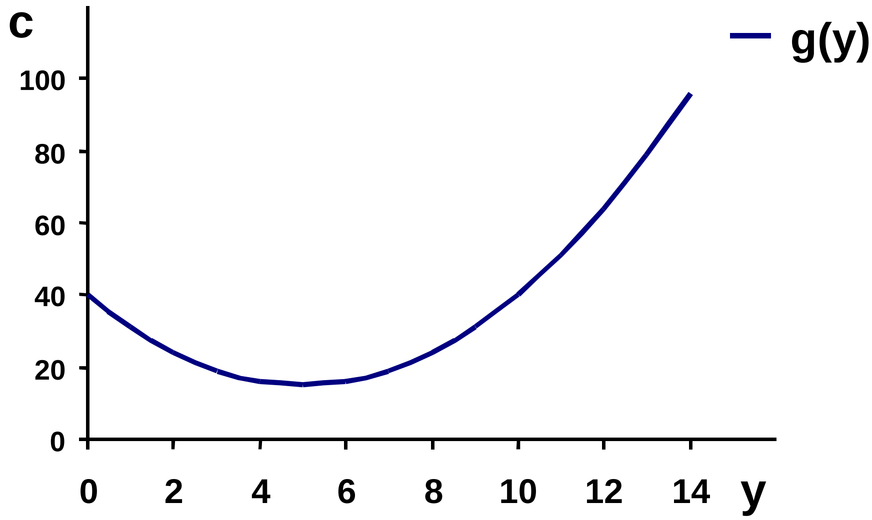
When graphing a function f(·) in 2 dimensions, it is customary to represent values in the domain of the function on the horizontal axis and corresponding values f(·) on the vertical axis. For the function y = f(x), x is represented on the horizontal axis and y is represented on the vertical axis. For the function, p = h(q), q is represented on the horizontal axis and p is represented on the vertical axis. While such a representation is customary, it is also acceptable to represent the domain on the vertical axis as long as the relationship is clear to the reader.

Examples



When x = 0, then f(x) = 8. When x = 4, f(x) = 0. When x = -2, f(x) = 12.

The Function $c = g(y) = 40 - 10y + y^2$



When y = 0, then c(y) = 40. When y = 5, c(y) = 15. When y = 10, c(y) = 40.

Linear functions

A linear function of a real variable x is given by

$$y = f(x) = ax + b, \quad a \text{ and } b \text{ are constant real numbers.}$$

The graph of a linear equation is a straight line. The number a is called the slope of the function and the number b is called the y-intercept. The y-intercept is the value of the function when x = 0. In the first graphical example in the previous section, y = f(x) = -2x + 8, the slope is -2 and the y-intercept is 8. In the equation p = h(q) = 100 - 4 q, the y-intercept is 100 and the slope is -4.

The slope of a linear function y = g(x) = ax + b measures the change (?) in g(x) divided by the change in x for any two points in the domain of g.

If the slope of a line is a positive number, then the graph of the function will slope upwards. If the slope of a line is a negative number, then the graph of the function will slope downwards. If the slope of a line is zero, then the graph of the function will be horizontal. If the slope of a line is positive infinity, then the graph of the function will be vertical.

Examples

1. Let $y = g(x) = 3x + 5$. Then consider a change in x from $x^0 = 2$ to $x^1 = 4$. The change in x is given by $x^1 - x^0$ or $4 - 2 = 2$. The function $g(\cdot)$ evaluated at $x^0 = 2$ is $g(2) = 11$ and the function $g(\cdot)$ evaluated at $x^1 = 4$ is $g(4) = 17$. The change in g or Δg is $g(4) - g(2) = 17 - 11 = 6$. The ratio of the changes is then given by

$$slope = \frac{g(x^1) - g(x^0)}{x^1 - x^0} = \frac{6}{2} = 3$$

If we interchange the designation of x^0 and x^1, we get the same answer.

Now consider a change in x from say $x^0 = -1$ to $x^1 = 2$. We can find the slope as follows

$$g(x) = 3x + 5$$

$$x^0 = -1, \quad x^1 = 2$$

$$g(-1) = -3 + 5 = 2$$

$$g(2) = -6 + 5 = 11$$

$$\Delta x = 2 - (-1) = 3$$

$$\Delta g(x) = 11 - 2 = 9$$

$$slope = \frac{g(x^1) - g(x^0)}{x^1 - x^0} = \frac{9}{2} = 3$$

For a linear function, the slope will always be the same no matter what points we consider.

2. Let $y = k(p) = 4p - 2$ where k is the function and p represents values in the domain of k. Then consider a change in p from $p^0 = 5$ to $p^1 = 4$. We can find the slope as follows

$$k(p) = 4p - 2$$

$$x^0 = 5, \quad x^1 = 4$$

$$k(5) = 20 - 2 = 18$$

$$k(4) = 16 - 2 = 14$$

$$\Delta p = 4 - 5 = -1$$

$$\Delta k(p) = 14 - 18 = -4$$

$$slope = \frac{k(p^1) - k(p^0)}{p^1 - p^0} = \frac{-4}{-1} = 4$$

The slope of a line given two distinct points on the line

If we know the values of x and h(x) for two different values of x, we can find the slope of the line without knowing the equation of the line. To see how this is done, consider two distinct points on a non-vertical straight line denoted by P = [x_1, $h(x_1)$] and Q = [x_2, $h(x_2)$]. Because the line is not vertical and P and Q are distinct, $x_1 \neq x_2$. The slope of the line is given by

$$a = slope = \frac{h(x_2) - h(x_1)}{x_2 - x_1}, \quad x_1 \neq x_2$$
$$= \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2 .$$

Of course, we can use different notation for the x variables, such as x^0 and x^1, or x_2 and x_3, etc.

Examples

Consider the following two points (2,4) and (5,10). We can find the slope as follows where we let $x_1 = 2$ and $x_2 = 5$.

$$a = slope = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2$$
$$= \frac{10 - 4}{5 - 2}$$
$$= \frac{6}{3} = 2.$$

Or consider the following two points (-2,4) and (6,8). We can find the slope as follows where we let $x^0 = -2$ and $x^1 = 6$.

$$a = slope = \frac{y^1 - y^0}{x^1 - x^0}, \quad x^1 \neq x^0$$
$$= \frac{8 - 4}{6 - (-2)}$$
$$= \frac{4}{8} = \frac{1}{2}.$$

Finding the equation for a line given a point on the line and the slope of the line

If we know the slope of a line and one point on the line we can find the equation for the line. To see how this is done, consider a point P = [x_1, $g(x_1)$] = [x_1, y_1] with a slope equal to a. Pick an arbitrary point on the line other than [x_1, y_1] and denote it as (x, y). Then write the formula for the slope of a line as follows.

$$slope = a = \frac{y - y_1}{x - x_1}, \quad x \neq x_1$$

Now solve the equation for either y as follows as a function of x as follows

$$\begin{aligned} a &= \frac{y - y_1}{x - x_1}, \quad x \neq x_1 \\ \Rightarrow y - y_1 &= a(x - x_1), \quad \textit{multiply both sides by } (x - x_1) \\ \Rightarrow y &= a(x - x_1) + y_1, \quad \textit{add } y_1 \textit{ to both sides} \\ &= ax + \left(y_1 - ax_1\right), \quad \textit{multiply out and combine terms.} \end{aligned}$$

The slope of the line is a and the y-intercept is (y_1 -ax_1). The second line of the above expression also turns out to be useful in many cases. We can rewrite it as follows

$$\begin{aligned} y - y_1 &= a(x - x_1) \\ \Rightarrow \Delta y &= a \Delta x. \end{aligned}$$

If we know the slope of a line and the change in x, we can find the change in y. Or if we know the change in x, the slope and one of the values for y, we can find the other value for y.

Examples

Suppose we know that x goes from 2 to 4 and that y is initially 6 or P = (2, 6). Suppose the slope of the line is 3. What is the value of y when x = 4. Here we let x_1 = 2, x = 4 and y_1 = 6. Using the above formula we obtain

$$\begin{aligned} y - y_1 &= a(x - x_1) \\ y - 6 &= 3(4 - 2) \\ \Rightarrow y - 6 &= 12 - 6 = 6 \\ \Rightarrow y &= 12. \end{aligned}$$

Suppose we know that y goes from 8 to 12 and that x is initially 2 or P = (2, 8). Suppose the slope of the line is -1. What is the value of x when y = 12. Here we let $x_1 = 2$, $y_1 = 8$ and y = 12. Using the above formula we obtain

$$y - y_1 = a(x - x_1)$$
$$12 - 8 = (-1)(x - 2)$$
$$\Rightarrow 4 = -x + 2$$
$$\Rightarrow x = -2$$

Consider the point (5,10) and a slope of 2. Here $x_1 = 5$ and $y_1 = 10$. What is the equation of this line?

$$a = \frac{y - y_1}{x - x_1}, \quad x \neq x_1$$
$$\Rightarrow 2 = \frac{y - 10}{x - 5}$$
$$\Rightarrow y - 10 = 2(x - 5), \quad \textit{multiply both sides by } (x - 5)$$
$$\Rightarrow y = 2(x - 5) + 10, \quad \textit{add } 10 \textit{ to both sides}$$
$$\Rightarrow y = 2x - 10 + 10, \quad \textit{multiply out}$$
$$\Rightarrow y = 2x$$

The slope is 2 and the y-intercept is 0.

Consider the point (6,8) and a slope of 1/2. Here $x_1 = 6$ and $y_1 = 8$. What is the equation of the line?

$$a = \frac{y - y_1}{x - x_1}, \quad x \neq x_1$$
$$\Rightarrow \frac{1}{2} = \frac{y - 8}{x - 6}$$
$$\Rightarrow x - 6 = 2(y - 8), \quad \textit{cross multiply}$$
$$\Rightarrow x - 6 = 2y - 16, \quad \textit{multiply out}$$
$$\Rightarrow 2y = x - 6 + 16, \quad \textit{add } 16 \textit{ to both sides}$$
$$\Rightarrow 2y = x + 10, \quad \textit{simplify}$$
$$\Rightarrow y = \frac{1}{2}x + 5, \quad \textit{divide both sides by } 2$$

The slope is 1/2 and the y-intercept is 5. We can check this out by substituting x = 6 in the equation

$$y = \frac{1}{2}x + 5$$

$$\Rightarrow y = \frac{1}{2}(6) + 5$$

$$= 3 + 5 = 8$$

Finding the equation of a line from two points on the line (point-point formula for a line)

If we are given two points on a line we can find the equation by first finding the slope and then using the point slope formula to find the equation for the line. Let the two points on the line be denoted [x_1, $g(x_1)$] = [x_1 , y_1] and [x_2, $g(x_2)$] = [x_2, y_2], $x_1 \neq x_2$. The slope is given by

$$a = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2.$$

If we substitute in the point-slope formula we obtain

$$\begin{aligned}
y - y_1 &= a(x - x_1) \\
&= \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) \\
\Rightarrow y &= \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) + y_1 \\
&= \left(\frac{y_2 - y_1}{x_2 - x_1}\right)x - \left(\frac{x_1(y_2 - y_1)}{x_2 - x_1}\right) + y_1 \\
&= \left(\frac{y_2 - y_1}{x_2 - x_1}\right)x + \left(\frac{-x_1y_2 + x_1y_1 + x_2y_1 - x_1y_1}{x_2 - x_1}\right) \\
&= \left(\frac{y_2 - y_1}{x_2 - x_1}\right)x + \left(\frac{x_2y_1 - x_1y_2}{x_2 - x_1}\right).
\end{aligned}$$

This formula looks complicated but the procedure is quite easy in practice. Consider two points (2,4) and (5,10). We can find the slope as follows where we let $x_1 = 2$ and $x_2 = 5$.

$$\begin{aligned}
a = slope &= \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2 \\
&= \frac{10 - 4}{5 - 2} \\
&= \frac{6}{3} = 2.
\end{aligned}$$

Now use the point slope formula with $(x_1, y_1) = (2, 4)$.

$$slope = a = \frac{y - y_1}{x - x_1}, \quad x \neq x_1$$

$$\Rightarrow 2 = \frac{y - 4}{x - 2}$$

$$\Rightarrow 2(x - 2) = y - 4$$

$$\Rightarrow y - 4 = 2x - 4$$

$$\Rightarrow y = 2x$$

Example

Consider two points (-2,4) and (10,10). We can find the slope as follows where we let $x_1 = -2$ and $x_2 = 10$.

$$a = slope = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2$$

$$= \frac{10 - 4}{4 - (-2)}$$

$$= \frac{10 - 4}{10 - 2}$$

$$= \frac{6}{12} = \frac{1}{2}.$$

Now use the point slope formula with $(x_2, y_2) = (10, 10)$.

$$slope = a = \frac{y - y_1}{x - x_1}, \quad x \neq x_1$$

$$\Rightarrow \frac{1}{2} = \frac{y - 10}{x - 10}$$

$$\Rightarrow (x - 10) = 2(y - 10)$$

$$\Rightarrow 2y - 20 = x - 10$$

$$\Rightarrow 2y = x + 10$$

$$\Rightarrow y = \frac{1}{2}x + 5$$