1 The Production Function: What Determines the Total Production of Goods and Services?

- An economy’s output of goods and services - its GDP - depends on (1) its quantity of inputs, called the factors of production, and (2) its ability to turn inputs into output, as represented by the production function.

- Factors of production are the inputs used to produce goods and services. The two most important factors of production are capital and labor.

  Capital is the set of tools that workers use (the construction worker’s crane, the accountant’s calculator, and your personal computer).

  Labor is the time (the number of hours) people spend working.

  We use the symbol $K$ to denote the amount of capital and the symbol $L$ to denote the amount of labor.

- Assume that the economy has a fixed amount of capital and a
fixed amount of labor. We write

\[ K = \bar{K} \]
\[ L = \bar{L} \]

The overbar means that each variable is fixed at some level.

We also assume here that the factors of production are fully utilized—that is, that no resources are wasted.

- Technology determines how much output is produced from given amounts of capital and labor. Economists express the available technology using a production function. Letting \( Y \) denote the amount of output, we write the production function as

\[ Y = F(K, L) \]

This equation states that output is a function of the amount of capital and the amount of labor.

- **Example 1 (linear production function)**

\[ F(K, L) = K + L \]

For example, suppose \( K = 2 \) and \( L = 5 \), then \( Y = F(K, L) = 2 + 5 = 7 \).

This production has the property that capital and labor are perfect substitutes. For example, if one reduces \( K \) by 1 unit but increases \( L \) by 1 unit, then output remains the same.
• Example 2

\[ F(K, L) = K \times L \]

This production function exhibits complementarity between capital and labor. To see this, suppose the initial capital and labor are \( K \) and \( L \). Suppose now one additional unit of labor becomes available. How much more output is produced because of the additional labor? Answer:

\[
K \times (L + 1) - K \times L = K \times 1
\]

which depends positively on \( K \). That is, a higher stock of capital makes the additional amount of labor more productive.

(Does this explain the higher labor productivity in the US than in Mexico?)

• Example 3 (A Cobb-Douglas Production Functions)

\[ F(K, L) = A \times K^\alpha L^{1-\alpha} \]

where \( A \) and \( \alpha \) are constants, \( A > 0 \) and \( 0 < \alpha < 1 \).

For example, \( A = 0.5, \alpha = 1/2 \). Let \( K = 4 \) and \( K = 9 \), then \( Y = 0.5 \times \sqrt{4} \times \sqrt{9} = 3 \).

• Example 4 (A Leontief production function)

\[ F(K, L) = \min\{L, K\} \]

This production function describes a technology with which in order to produce one unit of output, exactly (no more and no less) one unit of capital and one unit of labor is required. For example if \( L = 5 < K = 6 \), then \( Y = 5 \), the extra unit of capital is useless.
The production function reflects the current technology for turning capital and labor into output. If someone invents a better way to produce a good, the result is more output from the same amounts of capital and labor. Thus, technological change alters the production function.

Suppose today’s production function is \( Y = F(K, L) \), suppose tomorrow’s new technology is such that for any given amounts of capital and labour, output is doubled. Then tomorrow’s production function is

\[
Y = 2 \times F(K, L)
\]

Suppose today’s production function is \( Y = K + L \). Suppose tomorrow’s new technology makes tomorrow’s capital twice as productive as today’s capital. Then tomorrow’s production function is

\[
Y = 2 \times K + L
\]
2 Constant Returns to Scale

A production function has constant returns to scale if an increase of an equal percentage in all factors of production causes an increase in output of the same percentage. If the production function has constant returns to scale, then we get 10 percent more output when we increase both capital and labor by 10 percent.

- Mathematically, a production function has constant returns to scale if

\[ zF(k, L) = F(zK, zL) \]

for any positive number \( z \). This equation says that if we multiply both the amount of capital and the amount of labor by some number \( z \), output is also multiplied by \( z \).

- Example 1

\[ F(K, L) = K + L \]

- Example 3

\[ F(K, L) = AK^\alpha L^{1-\alpha} \]

- Example 4

\[ F(K, L) = \min\{K, L\} \]
3 The marginal product of Labor

The Marginal Product of Labor (MPL) is the extra amount of output that the economy gets from one extra unit of labor, holding the amount of capital fixed.

\[ MPL(K, L) = F(K, L + 1) - F(K, L) \]

That is, \( MPL \) is the difference between the amount of output produced using \( K \) units of capital and \((L + 1)\) units of labor and the amount of output produced using \( K \) units of capital and \( L \) units of labor.

Important to note: \( MPL \) depends on \((K, L)\), the levels of capital and labor at which \( MPL \) is computed.

- Let \( F(K, L) = K + L \). Then

\[ MPL(K, L) = F(K, L+1) - F(K, L) = (K+L+1)-(K+L) = 1 \]

Thus \( MPL \) is constant at one for all combinations of \( K \) and \( L \).

- Let \( F(K, L) = KL \). Then

\[ MPL(K, L) = K(L + 1) - KL = K. \]

In this example, \( MPL \) depends on \( K \) : it is higher when \( K \) is higher.
Let $F(K, L) = K^{1/2}L^{1/2} = \sqrt{KL}$. Then

$$MPL(K, L) = \sqrt{K(L+1)} - \sqrt{KL} = \sqrt{K} \left[ \sqrt{L + 1} - \sqrt{L} \right]$$

For example,

$$MPL(1, 0) = \sqrt{1}[\sqrt{0+1} - \sqrt{0}] = 1$$

$$MPL(1, 1) = \sqrt{1} [\sqrt{1+1} - \sqrt{1}] = 0.414$$

$$MPL(1, 2) = \sqrt{1}[\sqrt{2+1} - \sqrt{2}] = 1.73 - 1.41 = 0.32$$

Notice that here $M(1, L)$ decreases as $L$ increases.
4 The Law of Diminishing Marginal Product

- Many production functions have the property of Diminishing Marginal Product: Holding the amount of capital fixed, $MPL$ decreases as $L$ increases.

- Consider the production of bread at a bakery. As a bakery hires more labor, it produces more bread. As more workers are added to a fixed amount of capital, however, the $MPL$ falls. Fewer additional loaves of bread are produced because workers are less productive when the kitchen is more crowded. In other words, holding the size of the kitchen fixed, each additional worker adds fewer loaves of bread to the bakery’s output.

- Consider a farmer’s production of potatoes. He has a fixed amount of land that’s his capital $K$. Labor input $L$ here is the number of hours he spends working on his land. Output is higher as the farmer puts in more effort, but the productivity (additional output) associated with each additional hour the farmer puts in declines.

- Professor Wang has a computer that he uses as his capital. Professor Wang produces research by spending time with the computer. He is very productive at 9:00 in the morning, he is slower at 1:00 pm, and he feels his brain is useless at 6:30 in the afternoon.
5 The Marginal Product of Capital

The Marginal Product of Capital (MPK) is the extra amount of output that the economy gets from one extra unit of capital, holding the amount of labor fixed.

\[ MPL(K, L) = F(K + 1, L) - F(K, L) \]

That is, \( MPK \) is the difference between the amount of output produced using \( K + 1 \) units of capital instead of \( K \) units of capital.

- Let \( F(K, L) = K + L \). Then
  \[ MPL(K, L) = F(K+1, L) - F(K, L) = (K+1+L)-(K+L) = 1 \]
  Thus \( MPK \) is constant at one for all combinations of \( K \) and \( L \).

- Let \( F(K, L) = K \times L \). Then
  \[ MPL(K, L) = (K + 1)L - KL = L. \]
  In this example, \( MPK \) depends on \( L \) : it is higher when \( L \) is higher.
6 MPL and the demand for Labor

- Consider a firm which has a fixed amount of capital and wants to determine how many workers to hire. Let $P$ denote price and $W$ be the nominal wage (number of dollars paid to a worker).

- Suppose

\[ P \times MPL(K, L) > W \]

Then the firm would want to hire more workers.

- Suppose

\[ P \times MPL(K, L) < W \]

Then the firm would want to hire less workers.

- The firm’s equilibrium $L$ is determined by

\[ P \times MPL(K, L) = W \]

or

\[ MPL(K, L) = \frac{W}{P} \]

where $W/P$ is called the real wage.

Figure 3-4.

7 Case Studies

- We know that given $K$, the firm’s optimal amount of labor, $L^*$, is determined by the following equation
\[ MPL(K, L) = \frac{W}{P} \]

- Suppose it is now given that

\[ MPL(K, L) = K - L \]

where \( K = 10 \) is fixed. Suppose also \( W = 10, P = 2 \). Then \( L^* \) satisfies

\[ 10 - L^* = \frac{10}{2} \]

or

\[ L^* = 5 \]

- Suppose now the firm has just made some new capital investment and capital stock has increased from 10 to 15. Then the new \( L^* \) satisfies

\[ 15 - L^* = \frac{10}{2} \]

and \( L^* = 10 \).

- Suppose now the economy is heading into a recession and the firm has decided to cut capital stock from 15 to 5. Then the new \( L^* \) satisfies

\[ 5 - L^* = \frac{10}{2} \]

and \( L^* = 0 \). That is, the firm is shut down.