ECONOMICS 207
FALL 2010
PROBLEM SET 6
14^{th} April 2010

Problem 1: Consider the following function:

$$y = -2x^2 + 12x + 2$$

Find the critical values of x. Use the second derivative test to classify the critical points into local maximum and local minimum.

To find the critical points:

Step 1: $y' = -4x + 12$

Step 2: Set the derivative to zero. Solve to get the critical point

$y' = 0 \Rightarrow -4x + 12 = 0$

$\Rightarrow \boxed{x = 3}$.

Using the second derivative to figure out whether y is at a local max or local min at $x = 3$:

$y'' = -4$

$\because$ $\boxed{y'' < 0, \text{ there is a local max at } x = 3}$

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Problem 3: Consider the following function:

$$y = 2x^3 - \frac{17}{2}x^2 + 5x + 4$$

Find the critical values of x. Use the second derivative test to classify the critical points into local maximum and local minimum.

To find the critical points:

- $y' = 6x^2 - 17x + 5$
- $y' = 0 \Rightarrow 6x^2 - 17x + 5 = 0$

$$\Rightarrow (2x-5)(3x-1) = 0$$

$$\Rightarrow \boxed{x = 5/2,\ 1/3} \rightarrow \text{critical pts.}$$

Using the second derivative to figure out at which critical point we have a local max / min:

$$y'' = 12x - 17$$

$$y''(5/2) = 12\left(\frac{5}{2}\right) - 17 = 30 - 17 > 0$$

$$y''(1/3) = 12\left(\frac{1}{3}\right) - 17 = 4 - 17 < 0.$$

$\because y''(5/2) > 0$, y is at a local MIN at $x = 5/2$

$\because y''(1/3) < 0$, y is at a local MAX at $x = 1/3$

Problem 8: Consider the following function:

$$y = 5 - \frac{1}{4}x^4$$

a. Find the critical points of the function.

$y' = -x^3$

$y' = 0 \Rightarrow -x^3 = 0 \Rightarrow \boxed{x = 0}$

b. Is the function at a local maximum or a local minimum at the critical point.

$y'' = -3x^2$

$y''(0) = 0 \Rightarrow$ Second derivative test fails to give us any information as to whether there is a local max or min at $x = 0$. So we take help from higher derivative.

$y''' = -6x \qquad y'''(0) = 0$

$y'''' = -6 \leftarrow$ We stop here $\because$ we have encountered a non-zero #.

This is the 4^{th} derivative.

$\because$ 4 is an even number, we have a max or a min.

Moreover, $\because y'''' < 0$. $\therefore$ We have a local MAX at $x = 0$.

c.

$$z = e^{x^2 y}$$

$\dfrac{\partial z}{\partial x} = e^{x^2 y}(2xy)$

$= 2xy\, e^{x^2 y}$

$\dfrac{\partial z}{\partial y} = x^2 e^{x^2 y}$

$\dfrac{\partial^2 z}{\partial x^2} = 2y\left[e^{x^2 y} + 2x^2 y\, e^{x^2 y}\right]$

$\dfrac{\partial^2 z}{\partial y^2} = x^4 e^{x^2 y}$

$\dfrac{\partial^2 z}{\partial y \partial x} = 2x\left[e^{x^2 y} + x^2 y\, e^{x^2 y}\right]$

$\dfrac{\partial^2 z}{\partial x \partial y} = 2x\, e^{x^2 y} + x^2 \cdot (2xy)\, e^{x^2 y}$

$= 2x\left[e^{x^2 y} + x^2 y\, e^{x^2 y}\right]$

d.

$$z = 15x^{2/3}y^{3/5} - 2x - 4y$$

$\dfrac{\partial z}{\partial x} =$

$\dfrac{\partial z}{\partial y} =$

$\dfrac{\partial^2 z}{\partial x^2} =$

$\dfrac{\partial^2 z}{\partial y^2} =$

$\dfrac{\partial^2 z}{\partial y \partial x} =$

$\dfrac{\partial^2 z}{\partial x \partial y} =$

Problem 13: Find the critical points for each of the following functions. Use the Hessian Matrix to classify them as a local maxima, minima or saddle point.

a.

$$f(x_1, x_2) = 16x_1 + 12x_2 - x_1^2 - x_2^2$$

To find critical pts:

- $\frac{\partial f}{\partial x_1} = 16 - 2x_1$
- $\frac{\partial f}{\partial x_2} = 12 - 2x_2$

- $\frac{\partial f}{\partial x_1} = 0 \Rightarrow 16 - 2x_1 = 0$

  $\Rightarrow \boxed{x_1 = 8}$

- $\frac{\partial f}{\partial x_2} = 0 \Rightarrow 12 - 2x_2 = 0$

  $\Rightarrow \boxed{x_2 = 6}$

- Write down the Hessian

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$|H| = 4 - 0 > 0$

$\because |H| > 0$ we have a max/min

Furthermore, $\because H_{11}$ (1^{st} row & 1^{st} column element in H) is < 0,

it means, f is at a <u>max</u> at $x_1 = 8$ & $x_2 = 6$.

Problem 15: Consider a competitive firm with the following **cost function** and **output price**

$$C(y) = 1000 + 800y - 80y^2 + 4y^3$$

$$p = 748$$

where y represents output.

a. Write down the expression for the **profit function**. Find the **potential** levels of output that maximize profit (**i.e., the critical values** of y)

• First write the profit func^n: Profit = Revenue − Cost

Revenue ↓ comes from output

Cost ↓ comes from inputs

$$\therefore \pi = 748y - (1000 + 800y - 80y^2 + 4y^3)$$

So now we have profits as a function of y.
Next, we follow the usual procedure to figure out which value of y maximizes π.

c. Use the second derivative test to determine which of the levels of output from part (a) above actually maximizes profits.