Problem 1 (5 points each): Solve the following equations:

a. \(12x^7 + x^4 - x = 0\)

b. \(\left(\frac{11x^2 - 13x + 12}{2}\right)^{1/2} = 2x\)
c. \[ \frac{2x^2 - 9x + 9}{4x^2 - 9} = 0 \]

d. \[ \frac{3x^2 + 8x - 3}{3x - 1} = 2 - x \]
Problem 2 (10 points each): Solve the following systems of equations.

\[ 2x + 3y - z = 8 \]
\[ x - 2y + z = 1 \]
\[ 3x + y - z = 4 \]
b.

\[ 8x^{2/3}y^{-1/3} = 50 \]
\[ 15x^{-1/3}y^{1/2} = 24 \]
Problem 3 (4+4+2): Consider the following function

\[ y = x^4 - \frac{8}{3}x^3 - \frac{1}{2}x^2 + 2x - 4 \]

a. Find all the critical values of \( x \).

(hint: \( x = 1/2 \) is a critical value of \( x \))

b. Use the second derivative test to classify the critical points as local maximum, or minimum.

c. Find the potential points of inflection for this function.
Problem 4 (30 points): A firm’s production function is given by

\[ y = \text{output} = 40L + 20K - 2L^2 + 2LK - K^2 \]

where, \( y \), \( L \), and \( K \) represent output, labor and capital input respectively. The following information are also provided.

\[
\begin{align*}
\text{price per unit of output} & = 10 \\
\text{price per unit of labor} & = 120 \\
\text{price per unit of capital} & = 60
\end{align*}
\]

a. Write down the revenue function that represents revenue in terms of \( L \) and \( K \).

b. What is the cost function of the firm

c. What is the profit function of the firm

d. Find the critical values of \( L \) and \( K \) that can potentially maximize profits for the firm.

   Use the Hessian to verify that the critical values of \( L \) and \( K \) that you found actually do maximize profits. Give a reason why your conclusion is so.
Space
Problem 5 (40 points): Consider the following production function of a competitive firm which uses two inputs viz., labor and capital to produce its output,

\[ y = output = 20L + 30K - 2L^2 + LK - K^2 \]

where, \( L \) and \( K \) stand for labor and capital respectively. We are also given the following information:

\[ \text{price per unit of labor} = 30 \]
\[ \text{price per unit of capital} = 10 \]

However, the firm has a cost constraint. It cannot spend more than $300. How much \( L \) and \( K \) should the firm employ if it wants to maximize output given the cost constraint and the stated prices.

a. Write down an equation representing the constraint of the firm.

b. Write down the function that you want to maximize

c. Write down the Lagrangean function.

d. Find the critical values of \( L \), \( K \) and \( \lambda \) that can potentially maximize output for the firm given the cost constraint and the stated prices.

Use the Bordered Hessian to verify that the critical values of \( L \), \( K \) and \( \lambda \) that you found actually do maximize output for the firm given the cost constraint and the stated prices. Give a reason why your conclusion is so.
Space
Problem 6 (40 points): Consider a consumer who consumes two commodities: $x$ and $y$. The utility that he gets from consuming these commodities is given by:

$$ u = utility = 50x + 25y - 2x^2 + xy - 2y^2 $$

We are also given the following information:

- **price per unit of $x$** = 50
- **price per unit of $y$** = 40

The consumer’s **income is 470 dollars**. How much $x$ and $y$ should he consume if he wants to maximize his utility, given his income constraint and the stated prices.

Verify using the **Bordered hessian** the values of $x$, $y$ and $\lambda$ that you found actually do maximize the consumer’s utility, given his income constraint and the stated prices.
Space
Problem 7 (40 points): Consider a consumer who consumes two commodities: $x$ and $y$. The utility that he gets from consuming these commodities is given by:

$$u = utility = x^{2/5} y^{1/4}$$

We are also given the following information:

$$\text{price per unit of } x = 81$$

$$\text{price per unit of } y = 20$$

The consumer wants to achieve a target of 12 units of utility. How much $x$ and $y$ should he consume if he wants to achieve the target level of utility, at minimum cost given the stated prices.

Write down the Bordered Hessian matrix.

Verify using the Bordered hessian ($B$) that the values of $x$, $y$ and $\lambda$ that you found actually do minimize the consumer’s cost given his utility target and the stated prices.

[Hint: The values of $B_{11}$, $B_{13}$, $B_{22}$, $B_{23}$ evaluated at the critical values of $x$, $y$ and $\lambda$ are $\frac{243}{160}$, $\frac{3}{20}$, $\frac{5}{27}$, $\frac{1}{27}$ respectively, where $B_{ij}$ represents the element in the $i^{th}$ row and $j^{th}$ column of matrix $B$.]
Space