ECONOMICS 207
FALL 2006
PROBLEM SET 7

Problem 1. For each of the following systems of equations, find the solution vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ or $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ by appending the right-hand side vector to the coefficient matrix and performing row reduction.

(i)

$$\begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

$$\begin{bmatrix} -1 & 1 & -2 \\ 2 & -3 & 2 \end{bmatrix} \begin{matrix} R2 \to 2R1 + R2 \\ R1 \to -R1 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \begin{matrix} R1 \to R1 - R2 \\ R2 \to -R2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

So the solution to the system of equations is $(4, 2)$.

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$$\begin{pmatrix} 2 & -1 & 4 \\ 1 & 0 & 2 \\ 4 & -1 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 11 \\ 5 \\ 19 \end{pmatrix}$$

$$\begin{bmatrix} 2 & -1 & 4 & 11 \\ 1 & 0 & 2 & 5 \\ 4 & -1 & 7 & 19 \end{bmatrix} \text{Swap } R1 \text{ and } R2$$

$$\begin{bmatrix} 1 & 0 & 2 & 5 \\ 2 & -1 & 4 & 11 \\ 4 & -1 & 7 & 19 \end{bmatrix} \begin{matrix} R2 \to 2R1 - R2 \\ R3 \to 4R1 - R3 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix} R3 \to R2 - R3$$

$$\begin{bmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix} \begin{matrix} R1 \to R1 + 2R3 \\ R3 \to -R3 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Problem 2. Consider the following matrices.

$$A = \begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ -3 & -4 & -2 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & -3 & 2 \\ -2 & 5 & -2 \\ 4 & -11 & 7 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 7 \end{bmatrix}$$

(i) (a) Find the determinant of A.

$$\det(A) = \begin{vmatrix} -1 & 1 \\ 2 & -3 \end{vmatrix} = (-1)(-3) - (2)(1) = 1$$

(b) Find the inverse of A.

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{bmatrix} \begin{matrix} R2 \to \widetilde{2R1} + R2 \\ R1 \to -R1 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix} R2 \widetilde{\to -} R2$$

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & -1 \end{bmatrix} R1 \to \widetilde{R1} + R2$$

$$\begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & -2 & -1 \end{bmatrix}$$

$$So,\ A^{-1} = \begin{bmatrix} -3 & -1 \\ -2 & -1 \end{bmatrix}$$

(a) Find the determinant of C.

$$\det(C) = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = (1)(1) - (2)(0) = 1$$

(b) Find the inverse of C.

$$\begin{aligned} C^{-1} &= \frac{1}{\det(C)} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \end{aligned}$$

(a) Find the determinant of D.

$$\begin{aligned}
\det(D) &= \begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ -3 & -4 & -2 \end{vmatrix} \\
&= 1\begin{vmatrix} 5 & 2 \\ -4 & -2 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ -3 & -2 \end{vmatrix} + 1\begin{vmatrix} 2 & 5 \\ -3 & -4 \end{vmatrix} \\
&= [-10-(-8)] - 2\,[-4-(-6)] + [-8-(-15)] \\
&= -2 - 4 + 7 \\
&= 1
\end{aligned}$$

(b) Find the inverse of D.

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 5 & 2 & 0 & 1 & 0 \\ -3 & -4 & -2 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R2 \rightarrow \widetilde{-2R1} + R2 \\ R3 \rightarrow 3R1 + R3 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 2 & 1 & 3 & 0 & 1 \end{bmatrix} \begin{matrix} R1 \rightarrow \widetilde{-2R2} + R1 \\ R3 \rightarrow -2R2 + R3 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 5 & -2 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & 1 \end{bmatrix} R1 \rightarrow \widetilde{R1} - R3$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 0 & -1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & 1 \end{bmatrix}$$

$$So, \;\; D^{-1} = \begin{bmatrix} -2 & 0 & -1 \\ -2 & 1 & 0 \\ 7 & -2 & 1 \end{bmatrix}$$

(a) Find the determinant of F.

$$\begin{aligned}
\det(F) &= \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 7 \end{vmatrix} \\
&= 1\begin{vmatrix} 3 & 5 \\ 5 & 7 \end{vmatrix} - 2\begin{vmatrix} 2 & 5 \\ 3 & 7 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} \\
&= (21-25) - 2(14-15) + 3(10-9) \\
&= 1
\end{aligned}$$

(b) Find the inverse of F.

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 5 & 0 & 1 & 0 \\ 3 & 5 & 7 & 0 & 0 & 1 \end{bmatrix} \begin{array}{c} R2 \to \widetilde{-2R1} + 2R2 \\ R3 \to -3R1 + R3 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & -1 & -2 & -3 & 0 & 1 \end{bmatrix} \widetilde{R2 \to} -R2$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & -1 & -2 & -3 & 0 & 1 \end{bmatrix} \begin{array}{c} R1 \to \widetilde{-2R2} + R1 \\ R3 \to R2 + R3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 1 & -3 & 2 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{bmatrix} \begin{array}{c} R1 \to \widetilde{R1} + R3 \\ R2 \to R2 + R3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & -4 & 1 & 1 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{bmatrix} \widetilde{R3 \to} -R3$$

$$\begin{bmatrix} 1 & 0 & 0 & -4 & 1 & 1 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{bmatrix}$$

$$So, \;\; F^{-1} = \begin{bmatrix} -4 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$