ECONOMICS 207
FALL 2006
PROBLEM SET 9

Problem 1. Consider a consumer with the following utility function:

$$U(x_1, x_2) = x_1^{1/2} x_2^{1/3}$$

and facing the following prices for x_1 and x_2, respectively. w is the consumer's income.

$$p_1 = 2$$
$$p_2 = 3$$
$$w = 1000$$

Form the Lagrangian for the constrained maximization problem; then, find the first order conditions and solve for the optimal consumption bundle. (i.e., find the optimal values for x_1 and x_2) Remember, the Lagrangian is of the form: $\mathcal{L} = h(x_1, x_2) - \lambda [g(x_1, x_2)]$ where $h(x_1, x_2)$ is the objective function (the thing to be maximized) and $g(x_1, x_2)$ is the constraint (the budget constraint)

$$\mathcal{L} = x_1^{1/2} x_2^{1/3} - \lambda [2x_1 + 3x_2 - 1000]$$

FOC's:

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{1}{2} x_1^{-\frac{1}{2}} x_2^{\frac{1}{3}} - 2\lambda = 0 \;\Big|\; \frac{\partial \mathcal{L}}{\partial x_2} = \frac{1}{3} x_1^{\frac{1}{2}} x_3^{-\frac{2}{3}} - 3\lambda = 0 \;\Big|\; \frac{\partial \mathcal{L}}{\partial x_3} = 2x_1 + 3x_2 - 1000 = 0$$

Now with the first two first order conditions move the lambda terms to the right hand side of the equations, then take the ratio of the two equations.

$$\frac{\frac{1}{2} x_1^{-\frac{1}{2}} x_2^{\frac{1}{3}}}{\frac{1}{3} x_1^{\frac{1}{2}} x_3^{-\frac{2}{3}}} = \frac{2\lambda}{3\lambda}$$

$$\Rightarrow \quad x_2 = \frac{4}{9} x_1$$

Now substitute this into the third first order condition.

$$2x_1 + 3\left(\frac{4}{9} x_1\right) = 1000$$
$$x_1 = 300$$

So, then we find

$$x_2 = \frac{400}{3}$$

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Problem 2. Now check the second order condition on the bordered Hessian. Remember that the bordered Hessian in the constrained optimization problem written is given by:

$$H_B = \begin{bmatrix} -\frac{1}{4}x_1^{-\frac{3}{2}}x_2^{\frac{1}{3}} & \frac{1}{6}x_1^{-\frac{1}{2}}x_2^{-\frac{2}{3}} & 2 \\ \frac{1}{6}x_1^{-\frac{1}{2}}x_2^{-\frac{2}{3}} & -\frac{2}{9}x_1^{\frac{1}{2}}x_2^{-\frac{5}{3}} & 3 \\ 2 & 3 & 0 \end{bmatrix}$$

We evaluate H_B at our optimal values for x_1 and x_2 and find:

$$H_B = \begin{bmatrix} -.000246 & .000369 & 2 \\ .00369 & -.001106 & 3 \\ 2 & 3 & 0 \end{bmatrix}$$

This calculation was a little messy, but remember that when we are checking second order conditions, the sign is all that matters. So, we will use approximations here.

$$|H_B| \approx .011066 > 0$$

Hence, $\left(300, \frac{400}{3}\right)$ is a maximum.

Problem 3. Consider a consumer with the following utility function:

$$U(x_1, x_2) = x_1^{1/4} x_2^{1/5}$$

and facing the following prices for x_1 and x_2, respectively. w is the consumer's income.

$$p_1 = 1$$
$$p_2 = 2$$
$$w = \bar{w}$$

Form the Lagrangian for the constrained maximization problem; then, find the first order conditions and solve for the optimal consumption bundle. (i.e., find the optimal values for x_1 and x_2)

Remember, the Lagrangian is of the form: $\mathcal{L} = h(x_1, x_2) - \lambda [g(x_1, x_2)]$ where $h(x_1, x_2)$ is the objective function (the thing to be maximized) and $g(x_1, x_2)$ is the constraint (the budget constraint)

$$\mathcal{L} = x_1^{1/4} x_2^{1/5} - \lambda [x_1 + 2x_2 - \overline{w}]$$

FOC's:

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{1}{4} x_1^{-\frac{3}{4}} x_2^{\frac{1}{5}} - \lambda = 0 \;\Big|\; \frac{\partial \mathcal{L}}{\partial x_2} = \frac{1}{5} x_1^{\frac{1}{4}} x_3^{-\frac{4}{5}} - 2\lambda = 0 \;\Big|\; \frac{\partial \mathcal{L}}{\partial x_3} = x_1 + 2x_2 - \overline{w} = 0$$

Now with the first two first order conditions move the lambda terms to the right hand side of the equations, then take the ratio of the two equations.

$$\frac{\frac{1}{4} x_1^{-\frac{3}{4}} x_2^{\frac{1}{5}}}{\frac{1}{5} x_1^{\frac{1}{4}} x_3^{-\frac{4}{5}}} = \frac{\lambda}{2\lambda}$$

$$\Rightarrow \quad x_2 = \frac{2}{5} x_1$$

Now substitute this into the third first order condition.

$$x_1 + 2\left(\frac{2}{5} x_1\right) = \overline{w}$$

$$x_1 = \frac{5}{9} \overline{w}$$

So, then we find

$$x_2 = \frac{2}{9} \overline{w}$$

Problem 4. Now for this problem, write out the bordered Hessian and plug in the optimal values you found above. State what the condition is that gaurantees we found a maximum. (**Note: This means you are to write out the bordered Hessian, but I'm not asking you to calculate the determinant. Just tell me what you would look for if you were to calculate it.)** Remember that the bordered Hessian in the constrained optimization problem written is given by:

$$H_B = \begin{bmatrix} -\frac{3}{16}x_1^{-\frac{7}{4}}x_2^{\frac{1}{5}} & \frac{1}{20}x_1^{-\frac{3}{4}}x_2^{-\frac{4}{5}} & 1 \\ \frac{1}{20}x_1^{-\frac{3}{4}}x_2^{-\frac{4}{5}} & -\frac{4}{25}x_1^{\frac{1}{4}}x_2^{-\frac{9}{5}} & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

Evaluated at $x_1 = \frac{5}{9}\overline{w},\ x_2 = \frac{2}{9}\overline{w}$:

$$H_B = \begin{bmatrix} -\frac{3}{16}\left(\frac{5}{9}\overline{w}\right)^{-\frac{7}{4}}\left(\frac{2}{9}\overline{w}\right)^{\frac{1}{5}} & \frac{1}{20}\left(\frac{5}{9}\overline{w}\right)^{-\frac{3}{4}}\left(\frac{2}{9}\overline{w}\right)^{-\frac{4}{5}} & 1 \\ \frac{1}{20}\left(\frac{5}{9}\overline{w}\right)^{-\frac{3}{4}}\left(\frac{2}{9}\overline{w}\right)^{-\frac{4}{5}} & -\frac{4}{25}\left(\frac{5}{9}\overline{w}\right)^{\frac{1}{4}}\left(\frac{2}{9}\overline{w}\right)^{-\frac{9}{5}} & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

We could simplify this a little more, but since we are not going to calculate the determinant we will stop here. If we were to calculate the determinant, we would want to verify that $|H_B| > 0$.

Problem 5. Consider a consumer with the following utility function:

$$U(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$$

and facing the following prices for x_1 and x_2, respectively. w is the consumer's income.

$$p_1 = 1$$
$$p_2 = 2$$
$$w = 100$$

Form the Lagrangian for the constrained maximization problem; then, find the first order conditions and solve for the optimal consumption bundle. (i.e., find the optimal values for x_1 and x_2)

Remember, the Lagrangian is of the form: $\mathcal{L} = h(x_1, x_2) - \lambda [g(x_1, x_2)]$ where $h(x_1, x_2)$ is the objective function (the thing to be maximized) and $g(x_1, x_2)$ is the constraint (the budget constraint)

$$\mathcal{L} = x_1^{\alpha} x_2^{1-\alpha} - \lambda [x_1 + 2x_2 - 100]$$

FOC's:

$$\frac{\partial \mathcal{L}}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{1-\alpha} - \lambda = 0 \;\Big|\; \frac{\partial \mathcal{L}}{\partial x_2} = (1-\alpha) x_1^{\alpha} x_3^{-\alpha} - 2\lambda = 0 \;\Big|\; \frac{\partial \mathcal{L}}{\partial x_3} = x_1 + 2x_2 - 100 = 0$$

Now with the first two first order conditions move the lambda terms to the right hand side of the equations, then take the ratio of the two equations.

$$\frac{\alpha x_1^{\alpha-1} x_2^{1-\alpha}}{(1-\alpha) x_1^{\alpha} x_3^{-\alpha}} = \frac{\lambda}{2\lambda}$$
$$\Rightarrow \quad x_2 = \frac{(1-\alpha)}{\alpha} \frac{1}{2} x_1$$

Now substitute this into the third first order condition.

$$x_1 + 2\left(\frac{(1-\alpha)}{\alpha} \frac{1}{2} x_1\right) = 100$$
$$x_1 = 100\alpha$$

So, then we find

$$x_2 = \frac{1-\alpha}{2} 100$$

Now for this problem, write out the bordered Hessian and plug in the optimal values you found above. State what the condition is that gaurantees we found a maximum. (**Note: This means you are to write out the bordered Hessian, but I'm not asking you to calculate the determinant. Just tell me what you would look for if you were to calculate it.)** Remember that the bordered Hessian in the constrained optimization problem written is given by:

$$H_B = \begin{bmatrix} \alpha(\alpha-1)x_1^{\alpha-2}x_2^{1-\alpha} & \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} & 1 \\ \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} & -\alpha(1-\alpha)x_1^{\alpha}x_2^{-\alpha-1} & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

Evaluated at $x_1 = 100\alpha,\ x_2 = \frac{1-\alpha}{2}100$

$$H_B = \begin{bmatrix} \alpha(\alpha-1)(100\alpha)^{\alpha-2}\left(\frac{1-\alpha}{2}100\right)^{1-\alpha} & \alpha(1-\alpha)(100\alpha)^{\alpha-1}\left(\frac{1-\alpha}{2}100\right)^{-\alpha} & 1 \\ \alpha(1-\alpha)(100\alpha)^{\alpha-1}\left(\frac{1-\alpha}{2}100\right)^{-\alpha} & -\alpha(1-\alpha)(100\alpha)^{\alpha}\left(\frac{1-\alpha}{2}100\right)^{-\alpha-1} & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

If we were to verify the second order condition we would verify that $|H_B| > 0$

Problem 6. Consider a firm with production function $f(x_1, x_2)$, and who faces prices or $w_1,\ w_2$ for the inputs $x_1,\ x_2$ respectively. Find the profit maximizing quantities x_1 and x_2, and check the second order condition on the Hessian.

$$f(x_1, x_2) = 50x_1 + 30x_2 - 2x_1^2 + 2x_1x_2 - x_2^2$$
$$w_1 = 4$$
$$w_2 = 2$$
$$p = 1$$

$$\pi = \left(50x_1 + 30x_2 - 2x_1^2 + 2x_1x_2 - x_2^2\right) - 4x_1 - 2x_2$$
$$\pi = 46x_1 + 28x_2 - 2x_1^2 + 2x_1x_2 - x_2^2$$

FOC's:

$$\frac{\partial \pi}{\partial x_1} = 46 - 4x_1 + 2x_2 = 0$$
$$\frac{\partial \pi}{\partial x_2} = 28 + 2x_1 - 2x_2 = 0$$

Solving this system of equations we obtain $x_1 = 37,\ x_2 = 51$
To check the second order condition, we need to examine the Hessian matrix.

$$H = \begin{bmatrix} -4 & 2 \\ 2 & -2 \end{bmatrix}$$

$$\pi_{11} = -4 < 0$$
$$and\ |H| = 8 - 4 = 4 > 0$$

So $(37, 51)$ is a max.