ECONOMICS 207
SPRING 2005
EXAM 3

Problem 1 (24 points)**.**

For each of the following problems, find the critical points. For each critical point state whether the function is at a relative maximum, relative minimum, or otherwise.

a. $f(x) = 135x^2 - 3x^3$

The first derivative of the equation is $f'(x) = 270x - 9x^2$.

Setting this equal to zero we obtain:

$$\begin{aligned} 270x - 9x^2 &= 0 \\ \Rightarrow x(270 - 9x) &= 0 \\ \Rightarrow x &= 0 \text{ or } 270 = 9x \\ \Rightarrow x &= 0 \text{ or } x = 30 \end{aligned}$$

The second derivative of the function is $f''(x) = 270 - 18x$. By putting the first critical number $x = 0$ into the second derivative function we get, $f''(x) = 270 - 18(0) = 270 > 0$, which implies that $x = 0$ is a relative *minimum*. By putting the second critical number $x = 30$ into the second derivative function, we get $f''(x) = 270 - 18(30) = -270 > 0$ which implies that $x = 30$ is a relative *maximum*.

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b. $f(x) = 3x^5 - 5x^3$
The first derivative of the equation is $f'(x) = 15x^4 - 15x^2$. Setting this equal to zero we obtain:

$$\begin{aligned} 15x^4 - 15x^2 &= 0 \\ \Rightarrow x^2(15x^2 - 15) &= 0 \\ \Rightarrow x &= 0 \text{ or } 15x^2 = 15 \\ \Rightarrow x &= 0 \text{ or } x = -1 \text{ or } x = 1 \end{aligned}$$

The second derivative of the function is $f''(x) = 60x^3 - 30x$ By putting the first critical number $x = 0$ into the second derivative function, we get $f''(0) = 60(0)^3 - 30(0) = 0$ which is *indeterminate*. By putting the critical number $x = 1$ into the second derivative function, we get $f''(1) = 60(1)^3 - 30(1) = 30 > 0$ so $x = 1$ is a relative *minimum*. By putting the critical number $x = -1$ into the second derivative function, we get
$f''(-1) = 60(-1)^3 - 30(-1) = -30 < 0$ so $x = -1$ is a relative *maximum*.

Problem 2 (20 points)**.**

Find all the values of x for which the following firm maximizes profit where y is output, x is input, p is output price, and w is input price. Provide reasons for your decision based on concepts from differential calculus.
$y = 200x + 90x^2 - 3x^3,\ p = 2,\ w = 1048$

To solve for profit, first multiply output y by the output price p and subtract the input price w multiplied by input x:

$$\begin{aligned} Profit &= (output)(output\ price) - (input)(input\ price) \\ &= (y)(p) - (x)(w) \\ &= (200x + 90x^2 - 3x^3)(2) - (1048)(x) \\ &= -648x + 180x^2 - 6x^3 \end{aligned}$$

The derivative of the profit function is $f'(x) = -648 + 360x - 18x^2$. The critical points of the profit function are found by setting the derivative equal to zero and solving for x. We being

$$-648 + 360x - 18x^2 = 0$$

$$\Rightarrow x^2 - 20x + 36 = 0, \quad Now factor.$$

$$\Rightarrow (x - 2)(x - 18) = 0$$

$$\Rightarrow x = 2 \text{ or } x = 18$$

By doing this, we find critical points at $x = 2$ and at $x = 18$.

The second derivative of the profit function is $f''(x) = 360 - 36x$. When evaluated at the critical point $x = 2$ we get, $f''(2) = 360 - 36(2) = 288 > 0$ so there is a relative *minimum* when $x = 2$. And when the second derivative is evaluated at $x = 18$, we get $f''(18) = 360 - 36(18) = -288 < 0$ which indicates that there is a relative *maximum* at $x = 18$. The maximum at $x = 18$ gives us the level of input for which the firm will maximize the firm's profit (*Input*= 18).

The cost of production is equal to the input level multiplied by the price of the inputs:
$cost\ of\ production = (input)(input\ price) = (18)(1048) = 18864$

To find the optimal output level, we substitute $x = 18$ into the output function, $f(x) = 200x + 90x^2 - 3x^3$ and get $f(x) = 200(18) + 90(18)^2 - 3(18)^3 = 15264$ for our optimal level of output.

To find profit, we substitute $x = 18$ into the profit equation$-648x + 180x^2 - 6x^3$, to get,
$-648(18) + 180(18)^2 - 6(18)^3 = 11664$ as our profit

Problem 3 (10 points)**.**

Find the indefinite integral of each of the following functions with respect to the variable indicated. Write in the form F(x) + c.

a. $f(x) = 6x^2 + 2x - 5$, x
To find the indefinite integral of the function $f(x) = 6x^2 + 2x - 5$, we seek a function F that satisfies $F'(x) = 6x^2 + 2x - 5$ for all real x. So, from experience we can determine that $F(x) = 2x^3 + x^2 - 5x + C$ is a function with its derivative equal to $f(x) = 6x^2 + 2x - 5$.

$$f(x) = 6x^2 + 2x - 5 = \int 2x^3 + x^2 - 5x + C$$

b. $f(x) = 4x_1^{-1/2}x_2^{3/8} - 1$, x_1

$$f(x) = 4x_1^{-1/2}x_2^{3/8} - 1 = \int (8x_1^{1/2})(x_2^{3/8}) - x_1$$

This is similar to part a. above, except we are finding the indefinite integral with respect to x_1. This means that we only need to find the indefinite integral of the x_1 term while keeping the x_2 term constant.

Problem 4 (26 points)**.**

The cost function for a firm is a rule or mapping that tells the total cost of production of any output level produced by the firm. If the variable y represents the output of the firm, then the cost function is given by c(y). The consider the following competitive firm.

$$price \; = \; p \; = 319$$

$$cost \; = \; c(y) \; = 300 \; + \; 400y \; - \; 15y^2 \; + \; \frac{1}{3}y^3$$

a. What is the level of fixed cost for this firm?

   To find the fixed cost for the firm, we want to measure the cost to the firm when there is no output. To do this, we set output level equal to zero in the production cost function:

$$\begin{aligned} fixed\,cost \;&=\; c(0) \;=\; 300 \;+\; 400(0) \;-\; 15(0)^2 \;+\; \frac{1}{3}(0)^3 \\ &=\; c(0) \;=\; 300 \end{aligned}$$

   When $y = 0$ is plugged into the function, all of the terms go to zero with the exception of 300, which is the firm's fixed cost.

b. Write a function representing the revenue of the firm as a function of price and output level. Then substitute in the actual price the firm faces.

   To find the revenue of the firm, multiply the price of the output by the level of output.

$$Revenue = p \;\times\; y \;=\; 319 \times y$$

c. Write an equation that gives the variable cost of the firm as a function of the level of output.

   A firm's variable cost is the production expense that changes with the quantity of output produced. So variable cost is equal to the production cost function without the fixed cost.

$$Variable\ Cost = 400y \;-\; 15y^2 \;+\; \tfrac{1}{3}y^3$$

d. Write a function representing the profit of the firm as a function of output price and output level. Then substitute in the actual price the firm faces.

   Profit is equal to the output price x output level minus the cost of production. By doing so we get,

$$\begin{aligned} Profit &= py - c(y) \\ &= 319y - 300 - 400y + 15y^2 - \frac{1}{3}y^3 \\ &= -300 - 81y + 15y^2 - \frac{1}{3}y^3 \end{aligned}$$

This shows that the profit of the firm is a function of the output price and of the output level.

e. [6 points]

What is the profit maximizing level of output for this firm? Verify that the output level you choose is the only point a profit maximizing point.

To find the profit maximizing level, take the derivative of $Profit = -300 - 81y + 15y^2 - \frac{1}{3}y^3$ and find the critical points by setting the derivative equal to zero and solving for y.

$$\begin{aligned} Profit &= -300 - 81y + 15y^2 - \frac{1}{3}y^3 \\ \frac{dProfit}{dy} &= -81 + 30y - y^2 = 0 \\ \Rightarrow y^2 - 30y + 81 &= 0, \quad Now factor. \\ \Rightarrow (y - 3)(x - 27) &= 0 \\ \Rightarrow x &= 3 \text{ or } x = 27 \end{aligned}$$

When we do this, we find the critical points for the function to be $y = 3$ and $x = 27$.

To determine whether these critical points are maxima or minima or neither, we use a second derivative test. The second derivative of profit is given by

$$\frac{d^2Profit}{dy^2} = \pi''(y) = 30 - 2y$$

By plugging the critical numbers into the function we find that

$$\begin{aligned} \pi''(3) &= 24 \Rightarrow y = 3 \text{ is a minimum} \\ \pi''(27) &= -24 \Rightarrow y = 27 \text{ is a maximum} \end{aligned}$$

f. What is the total cost of the firm at this output level?

To find the cost of the firm at the output level $y = 27$, plug $y = 27$ into the production cost function.

$$\begin{aligned} Cost &= c(y) = 300 + 400y - 15y^2 + \frac{1}{3}y^3 \\ &= c(27) = 300 + 400(27) - 15(27)^2 + \frac{1}{3}(27)^3 = 6726 \end{aligned}$$

The total cost of production for the firm at the output level of $y = 27$ is equal to 6726.

g. Show that the revenue of this profit maximizing firm is $8,613.

To find the maximized revenue, take the price level multiplied by the optimal output level:

$$\begin{aligned} Maximized\ revenue &= (price)(maximized\ output\ level) \\ &= (319)(27) = 8613 \end{aligned}$$

h. What is the marginal cost of the firm at this output level?

To find the marginal cost of the firm, take the derivative of the production cost function to get:

$MC = c'(y) = 400 - 30y + y^2 = 319$

i. [4 points]

Using the marginal cost function from part h and revenue from part g, use integration to find producer surplus for this firm.

We can find the area under the marginal cost curve by integrating from zero to the output level. This will give

$$\begin{aligned} Variable\ Cost &= \int_0^{27} 400 - 30y + y^2\, dy \\ &= 400y - 15y^2 + \frac{1}{3}y^3 \,\Big|_0^{27} \\ &= (10,800 - 10,935 + 6,561) - 0 = 6,426 - 0 = 6,426 \end{aligned}$$

$$\begin{aligned} Producer\ Surplus &= (p)(y)\ -\ Variable\ Costs \\ &= (319)(27)\ -\ 6426 \\ &= 8613\ -\ 6426 \\ &= 2187 \end{aligned}$$

j. What is the profit level for this firm?

Profit is equal to the revenue minus the production cost. By substituting in the previously determined values for revenue and cost we can find the level of profit.

$$\begin{aligned} Profit &= Revenue\ -\ Cost \\ &= 8613 - 6726 \\ &= 1887 \end{aligned}$$

Problem 5 (18 points)**.**

Consider the following matrices.

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 1 & 3 \\ 3 & 4 & -4 \end{bmatrix}$$

$$D = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}, \quad F = \begin{bmatrix} 2 & 1 \\ -3 & -2 \\ 1 & 2 \end{bmatrix}$$

$$a = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

Compute the following

a. A + B

$$\begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} 7 & 5 \\ -1 & -1 \end{bmatrix}$$

To add matrices, simply add the corresponding entries together to solve the problem. For example, add the (1,1) entry of matrix A to the (1,1) entry of matrix B to get the (1,1) entry of matrix A+B (so, 4+3=7).

b. AB

$$\begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} 0 & \text{-1} \\ 1 & 0 \end{bmatrix}$$

When multiplying matrices, multiply the first row of matrix A times the first column of matrix B to get the (1,1) entry of AB. In the problem, multiply the first row of A [4 3] x the first column of B [-3 4]; this means that the (1,1) entry of AB is equal to [4 x 3 + 3 x -4] which equals zero. To find the (1,2) entry of AB, multiply the first row of A by the second column of B, [4 x 2 + 3 x -3] which equals -1.

c. AC

$$\begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -2 & 1 & 3 \\ 3 & 4 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 16 & 0 \\ 0 & 11 & 1 \end{bmatrix}$$

See the answer to part b.

d. AD

$$\begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

See the answer to part b.

e. C′

$$\begin{bmatrix} -2 & 1 & 3 \\ 3 & 4 & -4 \end{bmatrix}' = \begin{bmatrix} -2 & 3 \\ 1 & 4 \\ 3 & -4 \end{bmatrix}$$

To find C′, the first row of C becomes the first column of C′ and the second row of C becomes the second column of C′.

f. a′F

$$\begin{bmatrix} 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -3 & -2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \end{bmatrix}$$

See the answer to part b.

Problem 6 (2 points)**.**

What is the derivative of $e^{(2x^3+5x)}$ wiith respect to x?

To find the derivative, of $e^{(2x^3+5x)}$, take the derivative of the exponent (which is $(2x^3 + 5x)$) and multiply it by $e^{(2x^3+5x)}$ to get,

$$(6x^2 + 5)e^{(2x^3+5x)}$$