ECONOMICS 207
SPRING 2005
EXAM 4

Problem 1 (40 pts)**.** Consider the following matrices.

$$H = \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 14 \end{bmatrix}$$

a. Find the determinant of the matrix H.

The determinant of a 2 x 2 matrix is defined as $a_{11}a_{22} - a_{12}a_{21}$. So for matrix H, the determinant of matrix H is (2 x 4) - (1 x 6) = 2. The determinant of H is 2.

b. What is the cofactor of the element h_{11}?

To find the cofactor of a matrix entry, we use cofactor expansion. By doing this, we cross out the row and the column that the entry is in and form the remaining entries into a matrix. We then take the determinant of this new matrix and multiply it by $(-1)^{i+j}$, where the $(i + j)$ term represents the entry's position in the matrix (For example, if the entry was in row 1, column 2, then $(i + j)$ would equal $(1 + 2)$).

The cofactor of the element $h_{11} = (-1)^{i+j}$[4] $\rightarrow (-1)^{1+1}$[4] = 4.

Since this is only a 2 x 2 matrix, when we remove the ith row and jth column so there is only one remaining entry left. And because of the position of h_{11}, the $(-1)^{i+j}$ term is equal to 1 because $-1^2 = 1$.

c. What is the cofactor of the element h_{12}?

The cofactor of the element $h_{12} = -6$
See problem b above for explanation.

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d. What is the cofactor of the element h_{21}?

The cofactor of the element h_{21}=-1
See problem b above for explanation.

e. What is the cofactor of the element h_{22}?

The cofactor of the element h_{22}=2
See problem b above for explanation.

f. What is the cofactor matrix of the matrix H?

$$Cofactor\ matrix\ of\ H\ = \begin{bmatrix} 4 & -6 \\ -1 & 2 \end{bmatrix}$$

A cofactor matrix is a matrix showing the cofactors of the entries of the original matrix. So, the first entry of the cofactor matrix of H should be the cofactor of the first entry (h_{11}) in H. And in the $(1, 2)$ position of the cofactor matrix should be the cofactor of (h_{12}) of H, etc.

g. What is the adjoint matrix of the matrix H?

The adjoint matrix of H is the 2 x 2 matrix whose (i, j) entry is the (j, i) cofactor of H. This means that the adjoint matrix is the same as the cofactor matrix except that the row and columns are switched (the i^{th} row of the adjoint matrix is equal to the j^{th} column of the cofactor matrix). For a 3 x 3 matrix this would be

$$\text{cofactor matrix of H } = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix}$$

$$\text{adjoint matrix of H } = \begin{bmatrix} H_{11} & H_{21} & H_{31} \\ H_{12} & H_{22} & H_{32} \\ H_{13} & H_{23} & H_{33} \end{bmatrix}$$

So, by using the cofactors that we found above, we find that the cofactor matrix and adjoint matrix are as follows,

$$\text{Cofactor matrix of H} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ -1 & 2 \end{bmatrix}$$

and the adjoint matrix

$$= \begin{bmatrix} H_{11} & H_{21} \\ H_{12} & H_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -1 \\ -6 & 2 \end{bmatrix}$$

h. What is the inverse of the matrix H?

A 2 x 2 matrix has an inverse if and only if |H|$\neq 0$; since |H|$=\ 2$, H has an inverse. Since we have already found the adjoint matrix of H, we can multiply the adjoint matrix by $\frac{1}{|H|}$ to find the inverse of H.

The general form of an inverse for a 2 x 2 matrix is,

$$H^{-1} = \frac{1}{|H|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 & -1 \\ -6 & 2 \end{bmatrix} = \begin{bmatrix} 2 & \frac{-1}{2} \\ -3 & 1 \end{bmatrix}$$

i. [6 pts] Given the information you have from parts a-h, what is the solution to the equation

$$\begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 14 \end{pmatrix}$$

Since we have already found the inverse of H, we can use it to find the x-vector. To do this, we multiply H^{-1} by the matrix b to find the solution vector.

$$\begin{bmatrix} 2 & \frac{-1}{2} \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 14 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Therefore, the solution to $Hx = b$ is the vector $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

j. [10 pts] Using row reduction, find the inverse of the matrix H and the solution to the system of equations $H\,x \;=\; b$.

$$H = \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 14 \end{bmatrix}$$

To find the inverse of a matrix using row reduction, we make an augmented matrix with the original matrix on the left side and the identity matrix on the right side. We then use row reduction to find the inverse which will be given on the right side.

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 6 & 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & \frac{-1}{2} \\ 0 & 1 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & \frac{-1}{2} \\ -3 & 1 \end{bmatrix} = H^{-1}$$

To find the solution to $Hx = b$ we make an augmented matrix with matrix H on the left and matrix b on the right and use row reduction to solve for x.

$$H = \begin{bmatrix} 2 & 1 & 5 \\ 6 & 4 & 14 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{1}{2} & \frac{5}{2} \\ 0 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

The solution vector is the x-vector $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

The other method to solve both the inverse and the solution to the equation is to make an augmented matrix with matrix H on the left, the identity matrix (I_2) in the center, and matrix b on the right, and then use row reduction to find the answer. For an example of this method, see $problem\ 2j$.

Problem 2 (60 pts)**.** Consider the following matrices.

$$L = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 10 & 6 \\ 2 & 5 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 13 \\ 6 \end{bmatrix}$$

a. Find the determinant of the matrix L.

To find the determinant of a 3 x 3 matrix, you can use the basket-weaving method. The basket-weaving method is defined as $|\ A| = a_{11}a_{22}a_{33}$ + $a_{12}a_{23}a_{31}$ + $a_{13}a_{21}a_{32}$ - $a_{13}a_{22}a_{31}$ - $a_{11}a_{23}a_{32}$ - $a_{12}a_{21}a_{33}$. So, (1 x 10 x 5) + (3 x 6 x 2) + (2 x 3 x 5) - (2 x 10 x 2) - (1 x 6 x 5) - (3 x 3 x 5) = 50 + 36 + 30 - 40 - 30 - 45 = 1. The determinant of matrix L equals 1.

b. What is the cofactor of the element l_{11}?

The definition of the cofactor of an element of a 3 x 3 matrix is Ł$_{ij}$=$(-1)^{i+j}\ |\ M_{ij}\ |$ where M$_{ij}$ is the matrix (called the minor of the ijth element) formed by dropping the ith row and jth column of the matrix L. For explanation, see $problem\ 1.b$.

$$\begin{aligned} (-1)^{i+j}\ |\ M_{11}\ | &= (-1)^{1+1} \begin{vmatrix} 10 & 6 \\ 5 & 5 \end{vmatrix} \\ &= (-1)^{1+1}[(10\ \times\ 5)\ -\ (6\ \times\ 5)] \\ &= 20 \end{aligned}$$

c. What is the cofactor of the element l_{12}?

The definition of the cofactor of an element of a 3 x 3 matrix L is Ł$_{ij}$=$(-1)^{i+j}\ |\ M_{ij}\ |$.

$$(-1)^{1+2} \mid M_{12} \mid = (-1)^{1+2} \begin{bmatrix} 3 & 6 \\ 2 & 5 \end{bmatrix}$$
$$= (-1)^{1+2}[(3 \times 5) \ - \ (6 \times 2)]$$
$$= \ -3$$

d. What is the cofactor of the element l_{22}?

The definition of the cofactor of an element of a 3 x 3 matrix L is Ł$_{ij}$=$(-1)^{i+j} \mid M_{ij} \mid$.

$$(-1)^{2+2} \mid M_{22} \mid \ (-1)^{2+2} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$
$$= (-1)^{2+2}[(1 \times 5) \ - \ (2 \times 2)]$$
$$= 1$$

e. What is the cofactor of the element l_{23}?

The definition of the cofactor of an element of a 3 x 3 matrix L is Ł$_{ij}$=$(-1)^{i+j} \mid M_{ij} \mid$.

$$(-1)^{2+3} \mid M_{23} \mid = (-1)^{2+3} \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$$
$$= (-1)^{2+3}[(1 \times 5) \ - \ (3 \times 2)]$$
$$= 1$$

f. What is the cofactor of the element l_{33}?

The definition of the cofactor of a an element of 3 x 3 matrix L is Ł$_{ij}$=$(-1)^{i+j} \mid M_{ij} \mid$.

$$(-1)^{3+3} \mid M_{33} \mid = (-1)^{3+3} \begin{vmatrix} 1 & 3 \\ 3 & 10 \end{vmatrix}$$
$$= (-1)^{3+3}[(1 \times 10) \ - \ (3 \times 3)]$$
$$= 1$$

The cofactor matrix of the matrix L is given by

$$\begin{bmatrix} L_{11} & L_{12} & -5 \\ -5 & L_{22} & L_{23} \\ -2 & 0 & L_{33} \end{bmatrix} = \begin{bmatrix} 20 & -3 & -5 \\ -5 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix}$$

To find the cofactor matrix L, simply plug in the answers that were found above into the correct entry position in the matrix.

g. What is the adjoint matrix of the matrix L?

$$\begin{bmatrix} 20 & & -2 \\ & 1 & \\ & & \end{bmatrix} \to \begin{bmatrix} 20 & -5 & -2 \\ -3 & 1 & 0 \\ -5 & 1 & 1 \end{bmatrix}$$

The adjoint matrix of L is the 3 x 3 matrix whose (i, j) entry is the (j, i) cofactor of L. So by taking the previously determined cofactor matrix from above, we switch the ith and jth entries to obtain the adjoint matrix.

h. What is the inverse of the matrix L?

To find the inverse of L, we use the adjoint matrix and the determinant (both of which we have already found). Multiplying the adjoint matrix by $\frac{1}{|H|}$ will get us the inverse of L.

$$\frac{1}{1}\begin{bmatrix} 20 & -5 & -2 \\ -3 & 1 & 0 \\ -5 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 20 & -5 & -2 \\ -3 & 1 & 0 \\ -5 & 1 & 1 \end{bmatrix}$$

i. [6 pts] Given the information you have from parts a-h, what is the solution to the equation

$$\begin{pmatrix} 1 & 3 & 2 \\ 3 & 10 & 6 \\ 2 & 5 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 13 \\ 6 \end{pmatrix}$$

To find the solution to the equation, we multiply L^{-1} by the matrix b.

$$\begin{bmatrix} 20 & -5 & -2 \\ -3 & 1 & 0 \\ -5 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 13 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

The vector [3,1,-1] is the solution to the equation.

j. [14 pts] Using row reduction, find the inverse of the matrix L and the solution to the system of equations $L\,x = b$.

$$L = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 10 & 6 \\ 2 & 5 & 5 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 13 \\ 6 \end{bmatrix}$$

$$\tilde{L} = \begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 & 4 \\ 3 & 10 & 6 & 0 & 1 & 0 & 13 \\ 2 & 5 & 5 & 0 & 0 & 1 & 6 \end{bmatrix}$$

To find the inverse of L and the solution to $Lx = b$ simultaneously, we create an augmented matrix with the original matrix L, the identity matrix I_3, and the matrix b (from left to right) and will call it $\tilde{L}$. Once we have assembled $\tilde{L}$, we use row reduction to find the inverse and the solution set.

$$\tilde{L} = \begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 & 4 \\ 3 & 10 & 6 & 0 & 1 & 0 & 13 \\ 2 & 5 & 5 & 0 & 0 & 1 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 & 1 & 0 & 1 \\ 0 & -1 & 1 & -2 & 0 & 1 & -2 \end{bmatrix} \rightarrow$$

$$\rightarrow \begin{bmatrix} 1 & 3 & 2 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 & 1 & 0 & 1 \\ 0 & 0 & 1 & -5 & 1 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 10 & -3 & 0 & 1 \\ 0 & 1 & 0 & -3 & 1 & 0 & 1 \\ 0 & 0 & 1 & -5 & 1 & 1 & -1 \end{bmatrix} \rightarrow$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 20 & -5 & -2 & 3 \\ 0 & 1 & 0 & -3 & 1 & 0 & 1 \\ 0 & 0 & 1 & -5 & 1 & 1 & -1 \end{bmatrix}$$

From the solved $\tilde{L}$, we know that the inverse of matrix L is,

$$L^{-1} = \begin{bmatrix} 20 & -5 & -2 \\ -3 & 1 & 0 \\ -5 & 1 & 1 \end{bmatrix}$$

And the solution set to $Lx = b$ is equal to,

$$x = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$