ECONOMICS 207
SPRING 2005
EXAMINATION 5

Problem 1 (20 points)**.** The cost function for a firm is a rule or mapping that tells the total cost of production of any output level produced by the firm. If y represents the output of the firm, then the cost function is given by c(y). Consider the following competitive firm.

$$price \;=\; p \;= 264$$

$$cost \;=\; c(y) \;= 100 \;+\; 300y \;-\; 10y^2 \;+\; \frac{1}{3}y^3$$

a. Write a function representing the profit of the firm as a function of the given output price and output level.
To find profit, take the revenue of the firm and subtract the firm's cost.

$$\begin{aligned} \pi &= R - c(y) \\ &= (p)(y) - c(y) \\ &= (264)y - [100 \;+\; 300y \;-\; 10y^2 \;+\; \frac{1}{3}y^3] \\ &= (264)y - 100 \;-\; 300y \;+\; 10y^2 \;-\; \frac{1}{3}y^3 \\ &= -36y - 100 \;+\; 10y^2 \;-\; \frac{1}{3}y^3 \end{aligned}$$

b. What is the profit maximizing level of output for this firm? Verify that the output level you choose is the profit maximizing point.
To find the profit maximizing level of the firm take the derivative of the profit function and set it equal to zero. Then solve for y.

$$\pi \;=\; -36y - 100 \;+\; 10y^2 \;-\; \frac{1}{3}y^3$$

$$\begin{aligned} \pi' &= -y^2 + 20y - 36 = 0 \\ \Rightarrow & \quad (-y+2)(y-18) = 0 \\ \Rightarrow y &= 2, \text{ or } y = 18 \end{aligned}$$

Because there are critical values ay y equal to 2 and y equal to 18, we need to determine which critical point is a maximum and which is a minimum. To do this, we take the second derivative of the profit function and plug in the critical points.

Date: April 4, 2006.

$Second\ derivative: \ \pi'' = 20 - 2y$
To test the critical point at 2, $\pi'' = 20 - 2(2) = 16 > 0$. Since $\pi''(2)$ is greater than 0, it is a minimum. To test the critical point at 18, $\pi'' = 20 - 2(18) = -16 < 0$. Since $\pi''(18)$ is less than 0, it is a maximum.

To find the profit, take revenue minus the cost at the profit maximizing level of output.

$$\begin{aligned}\pi(y) &= -36y - 100 + 10y^2 - \frac{1}{3}y^3 \\ &= -36(18) - 100 + 10(18)^2 - \frac{1}{3}(18)^3 \\ &= 548\end{aligned}$$

c. What is the marginal cost of the firm at this output level?
The marginal cost is equal to the derivative of the cost function,

$$\begin{aligned}cost &= c(y) = 100 + 300y - 10y^2 + \frac{1}{3}y^3 \\ \Rightarrow c'(y) &= 300 - 20y + y^2 = MC \\ \Rightarrow MC &= 300 - 20(18) + (18)^2 = 264\end{aligned}$$

The marginal cost of the firm at the optimal output level is 264.

d. What is the variable cost of production at this output level?
The variable cost is the firm's total cost minus the fixed costs and is dependent on the output level.

$$\begin{aligned}VC &= c(y) - FC \\ &= (\ 100 + 300y - 10y^2 + \frac{1}{3}y^3) - 100 \\ &= 300y - 10y^2 + \frac{1}{3}y^3 = VC\end{aligned}$$

Then to find the VC at the output level of y=18, insert 18 into the VC function,

$$\begin{aligned}VC(y) &= 300y - 10y^2 + \frac{1}{3}y^3 \\ VC(18) &= 300(18) - 10(18)^2 + \frac{1}{3}(18)^3 \\ &= 4104\end{aligned}$$

Variable cost is 4104.

Problem 2 (20 points)**.** Consider a competitive firm with the following production function where y represents the level of output of the firm and x represents the level of the single variable input.

$$y = f(x) = 100x + 60x^2 - 2x^3$$

The firm faces an output price of p = 10 and an input price of w = 3160.

a. Write a function representing the profit of the firm as a function of the given output and input prices and the input level.

Since profit is equal to revenue minus cost we get,

$$\begin{aligned}
\pi &= pf(x) - wx \\
&= (10)(100x + 60x^2 - 2x^3) - wx \\
&= (1000x + 600x^2 - 20x^3 - 3160x \\
&= -2160x + 600x^2 - 20x^3
\end{aligned}$$

b. What is the profit maximizing level of input for this firm? Verify that the input level you choose is the profit maximizing point.

To find the profit maximizing level of input for the firm, set the derivative of the profit equation equal to 0 and solve for x.

$$\begin{aligned}
\pi &= -2160x + 600x^2 - 20x^3 \\
\pi' &= -2160 + 1200x - 60x^2 = 0 \\
&\Rightarrow -36 + 20x - x^2 = 0 \\
&\Rightarrow (x-2)(-x+18) = 0 \\
\Rightarrow x &= 2, \text{ or } x = 18
\end{aligned}$$

The critical points of x are 18 and 2, and to check for local extreme points, we need to find the second derivative of the profit equation and plug in our values of x. First we check for x=18:

$$\begin{aligned}
\pi''(x) &= 1200 - 120x \\
&= 10(120 - 12x) = 0 \\
&= 10(120 - 12(18)) = -960 < 0
\end{aligned}$$

Because π''(18)= -960 is less than 0, x=18 is a maximum. Now we check x=2:

$$\begin{aligned}
\pi''(x) &= 1200 - 120x \\
&= 10(120 - 12x) = 0 \\
&= 10(120 - 12(2)) = 960 > 0
\end{aligned}$$

Because π''(2) = 960 is greater than 0, x=2 is a minimum.

Therefore, x=18 is the optimal input level for the firm.

To find the maximized profit, take the revenue minus the cost to get,

$$\pi = R - C = (-2160x \ + \ 600x^2 \ - \ 20x^3) = [-2160(18) \ + \ 600(18)^2 \ - \ 20(18)^3] = 38,880$$

c. What is the level of output of the firm at the profit maximizing input level?

To find the level of output of the firm at the profit maximizing level, we insert x=18 into the production function.

$$\begin{aligned} Optimal\ output \ : \ f(x) \ &= \ 100(x) \ + \ 60(x)^2 \ - \ 2(x)^3 \\ f(18) \ &= \ 100(18) \ + \ 60(18)^2 \ - \ 2(18)^3 \\ f(18) \ &= \ 9576 \end{aligned}$$

The optimal level of output is 9576.

d. What is the marginal product (MP_x) of the variable input at the profit maximizing input level?
The partial derivative of $f(x)$ with respect to x is the marginal product of x. So we obtain

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial(100(x) \ + \ 60(x)^2 \ - \ 2(x)^3)}{\partial x} \ = \ 100 \ + \ 120x \ - \ 6x^2 \\ &= 100 \ + \ (120)(18) \ - \ (6)(18^2) \\ &= 100 \ + \ 2160 \ - \ 1944 \\ &= 316 \end{aligned}$$

The optimality conditions for the profit maximization problem can be interpreted as

$$\begin{aligned} \frac{\partial \pi}{\partial x} &= p \frac{\partial f(x)}{\partial x} \ - \ w \ = \ 0 \\ \Rightarrow \ p MP_x \ &= w \end{aligned}$$

And so we obtain

$$\frac{\partial f(x)}{\partial x} \ = \ \frac{w}{p} \ = \ \frac{3160}{10}$$

$$MP_x \ = \ 316$$

e. Verify that $pMP_x = w$ at the profit maximizing input level.

To verify that $pMP_x = w$, we plug in the respective values.

$pMP_x = w \rightarrow (10)(316) = 3160.$

Problem 3 (60 points)**.** For each of the following problems, write an equation that represents profit as a function of the two inputs x_1 and x_2. Output price is represented by p, the price of the first input by w_1 and the price of the second input by w_2. Write it in the form π = $pf(x_1,x_2)-w_1x_1-w_2x_2$ and then simplify the expression. Then find critical values of x_1 and x_2. Verify that profit is maximized at these critical values.

a.

$$\begin{aligned} f(x_1, x_2) &= 40x_1 + 20x_2 - 2x_1^2 + 2x_1x_2 - x_2^2 \\ p &= 3 \\ w_1 &= 30, \qquad w_2 = 18 \end{aligned}$$

First write the profit equation

$$\pi = p(40x_1 + 20x_2 - 2x_1^2 + 2x_1x_2 - x_2^2) - w_1x_1 - w_2x_2$$

First, we must find the partial derivative of π with respect to x_1, and then the partial derivative of π with respect to x_2.

We obtain

$$\begin{aligned} \frac{\partial \pi}{\partial x_1} &= p(40 - 4x_1 + 2x_2) - w_1 \\ \frac{\partial \pi}{\partial x_2} &= p(20 + 2x_1 - 2x_2) - w_2 \end{aligned}$$

Then, by setting the two equations equal to zero, we can solve for the critical values of x_1 and x_2.

$$\frac{\partial \pi}{\partial x_1} = p(40 - 4x_1 + 2x_2) - w_1 = 0 \tag{1a}$$

$$\frac{\partial \pi}{\partial x_2} = p(20 + 2x_1 - 2x_2) - w_2 = 0 \tag{1b}$$

Add the first equation in 1 to the second equation to obtain

$$\begin{array}{cccccccccccc} p & (40 & + & -4x_1 & + & 2x_2) & + & -w_1 & + & 0w_2 & = & 0 \\ p & (20 & + & 2x_1 & + & -2x_2) & + & 0w_1 & + & -w_2 & = & 0 \\ \hline p & (60 & + & -2x_1 & + & 0x_2) & + & -w_1 & + & -w_2 & = & 0 \end{array}$$

$$\begin{aligned} &\Rightarrow p(60 - 2x_1) = w_1 + w_2 \\ &\Rightarrow -2px_1 = w_1 + w_2 - 60p \\ &\Rightarrow x_1 = \frac{60p - w_1 - w_2}{2p} \end{aligned} \tag{2}$$

Now substitute x_1 into the first equation to obtain

$$\begin{aligned} p\left(40 \, + \, -4\left(\frac{60p \, - \, w_1 \, - \, w_2}{2p}\right) \, + \, 2x_2\right) \, - \, w_1 \, &= 0 \\ \Rightarrow \, 40p \, - \, 120p \, + \, 2w_1 \, + \, 2w_2 \, + \, 2px_2 \, - \, w_1 \, &= 0 \\ \Rightarrow \, 2px_2 \, &= 80p \, - \, w_1 \, - \, 2w_2 \\ \Rightarrow \, x_2 \, &= \frac{80p \, - \, w_1 \, - \, 2w_2}{2p} \end{aligned} \tag{3}$$

By plugging in the values of p, w_1, and w_2 into the optimal input level equations in 2 and 2, we get the optimal input levels: $x_1 = 22$ and $x_2 = 29$.

The profit function has a local maximum at the point x^* if $\frac{\partial^2 \pi}{\partial x_1^2}(x^*) \, < \, 0 \, and$ $\frac{\partial^2 \pi}{\partial x_1^2} \frac{\partial^2 \pi}{\partial x_2^2} \, - \, \left[\frac{\partial^2 \pi}{\partial x_1 \, \partial x_2}\right]^2 \, > \, 0 \; at \; x^*$. We can say this in another way as follows. We can verify that x^* is a maximum if we set up a Hessian Matrix such as:

$$\begin{vmatrix} \frac{\partial^2 \pi(x^*)}{\partial x_1^2} & \frac{\partial^2 \pi(x^*)}{\partial x_1 \, \partial x_2} \\ \frac{\partial^2 \pi(x^*)}{\partial x_2 \, \partial x_1} & \frac{\partial^2 \pi(x^*)}{\partial x_2^2} \end{vmatrix}$$

If the determinant of the Hessian Matrix (discriminant) is greater than 0 and the (1,1) entry is negative, then π has a local maximum. The first entry in the Hessian Matrix is found by taking the derivative of π with respect to x_1 two times, the (1,2) entry is determined by taking the derivative of π with respect to x_2 and then with respect to x_1. The (2,1) entry is found by taking the derivative of π with respect to x_1 and then x_2, and the (2,2) entry is found by taking the derivative of of π with respect to x_2 twice.
When we take the second derivative with respect to x_1 we get -4, when we first take the derivative of with respect to x_1 and then with respect to x_2 we get 2, when we take the derivative first with respect to x_2 and then with respect to x_1 we get 2, and finally when we take the derivative with respect to of x_2 twice we obtain, -2. So our Hessian Matrix is:

$$\begin{vmatrix} -4 & 2 \\ 2 & -2 \end{vmatrix} \, = \, 4 \, > \, 0 \tag{4}$$

Because the detrminant of the Hessian Matrix is greater than 0, and because the (1,1) is negative, there is local maximum with $x_1 = 22$ and $x_2 = 29$.

To find the optimal output level with the optimal input levels, insert $x_1 = 22$ and $x_2 = 29$ into the production function to obtain, $f(22, 29) \, = \, 40(22) + 20(29) - 2(29)^2 + 2(22)(29) - (29)^2 \, = \, 927$. So 927 is the firm's optimal outptut level.

Given the input level and output level of the firm, we can determine the firm's profit. Profit is equal to revenue minus the cost, or:

$$\begin{aligned} \pi \, &= \, R \, - \, C \\ \pi \, &= \, (price)(output\ level) \, - \, (price\ of\ inputs)(input\ level) \\ \pi \, &= \, (3)(927) \, - \, (30)(22) - (18)(29) \, = \, 1599 \end{aligned}$$

The firm's profit is equal to 1599.

b.

$$\begin{aligned} f(x_1, x_2) &= x_1^{1/4} x_2^{1/3} \\ p &= 48 \\ w_1 &= 3, \qquad w_2 \ = \ 8 \end{aligned}$$

The determinant of the Hessian of the profit equation is $\frac{5}{64} = 0.08$.

$$\pi \ = \ 48 x_1^{1/4} x_2^{1/3} \ - \ 3x_1 \ - \ 8x_2$$

First, we need to solve for the critical points x_1 and x_2. To do this, take the partial derivatives of π with respect to x_1 and x_2, set them equal to zero, and solve for x_1 and x_2.

$$\begin{aligned} \frac{\partial \pi}{\partial x_1} &= \quad 12 x_1^{\frac{-3}{4}} x_2^{\frac{1}{3}} \ - \ 3 \ = \ 0 \\ \frac{\partial \pi}{\partial x_2} &= \quad 16 x_1^{\frac{1}{4}} x_2^{\frac{-2}{3}} \ - 8 \ = \ 0 \end{aligned} \tag{5}$$

Solve the first equation in 5 for x_1 as follows

$$\begin{aligned} 12 x_1^{\frac{-3}{4}} x_2^{\frac{1}{3}} \ - \ 3 \ &= 0 \\ \Rightarrow \ x_1^{\frac{-3}{4}} x_2^{\frac{1}{3}} \ &= \frac{3}{12} \ = \ \frac{1}{4} \\ \Rightarrow \ x_1^{\frac{-3}{4}} \ &= \frac{1}{4} x_2^{\frac{-1}{3}} \\ \Rightarrow \ x_1 \ &= \left(\frac{1}{4} x_2^{\frac{-1}{3}} \right)^{-4/3} \ = \ \left(\frac{1}{4} \right)^{\frac{-4}{3}} x_2^{4/9} \end{aligned} \tag{6}$$

Now substitute the answer from equation 6 into the second equation of 5 to obtain

$$\begin{aligned} 16 \left(\left(\frac{1}{4} \right)^{\frac{-4}{3}} x_2^{4/9} \right)^{\frac{1}{4}} x_2^{\frac{-2}{3}} \ &= 8 \\ \Rightarrow \ \left(\frac{1}{4} \right)^{-1/3} x_2^{1/9} x_2^{\frac{-2}{3}} \ &= \frac{1}{2} \\ \Rightarrow \ x_2^{\frac{-5}{9}} \ &= \frac{1}{2} \left(\frac{1}{4} \right)^{1/3} \\ &= \left(\frac{1}{2} \left(\frac{1}{4} \right)^{1/3} \right)^{\frac{-9}{5}} \\ &= 2^{\frac{9}{5}} \left(4^{-1/3} \right)^{\frac{-9}{5}} \ = \ 2^{\frac{9}{5}} \left(2^{-2/3} \right)^{\frac{-9}{5}} \\ &= 2^{\frac{9}{5}} \, 2^{18/15} \ = \ 2^{\frac{27}{15}} \, 2^{18/15} \ = \ 2^{\frac{45}{15}} \ = \ 2^3 \ = \ 8 \end{aligned} \tag{7}$$

Substituting back into equation 6 we obtain

$$
\begin{aligned}
x_1 &= \left(\frac{1}{4}\right)^{\frac{-4}{3}} (8)^{4/9} \\
&= (2^2)^{\frac{4}{3}} (2^3)^{\frac{4}{9}} \\
&= 2^{\frac{8}{3}}\, 2^{\frac{12}{9}} \\
&= 2^{\frac{24}{9}}\, 2^{\frac{12}{9}} \\
&= 2^{\frac{36}{9}} = 2^4 = 16
\end{aligned} \tag{8}
$$

As the determinant of the Hessian Matrix is greater than zero, we just need to check that $\frac{\partial^2 \pi}{\partial x_1^2}$ is less than zero to verify that profit at $x_1 = 16$ and $x_2 = 8$ is a maximum.

The second derivative of π evaluated at $x_1 = 16$ and $x_2 = 8$ is

$$
\begin{aligned}
\frac{\partial^2 \pi}{\partial x_1^2} &= -9 x_1^{\frac{-7}{4}} x_2^{\frac{1}{3}} \\
&= (-9)(16)^{\frac{-7}{4}} (8)^{\frac{1}{3}} \\
&= (-9)(2)^{-7}(2) = \frac{-9}{64}
\end{aligned}
$$

To find the optimal output level, plug the values for x_1 and x_2 into the production function.

$$
f(x_1, x_2) = x_1^{1/4} x_2^{1/3}
$$

$$
\begin{aligned}
f(16, 8) &= 16_1^{1/4} 2_2^{1/3} \\
&= (2)(2) \\
&= 4
\end{aligned}
$$

The optimal output level is 4.

To find the profit, we take the revenue minus the cost.

$$
\begin{aligned}
\pi &= R - C \\
&= (price)(output\ level) - (input\ level)(input\ price) \\
&= (p)(output\ level) - [(w_1)(x_1) + (w_2)(x_2)] \\
&= (48)(4) - [(16)(3) + (8)(8)] \\
&= 192 - 112 \\
&= 80
\end{aligned}
$$

The firm's profit is 80.