Problem 1. The cost function for a firm is a rule or mapping that tells the total cost of production of any output level produced by the firm. If \( y \) represents the output of the firm, then the cost function is given by \( c(y) \). Consider the following competitive firm.

\[
\text{price} = p = 409 \\
\text{cost} = c(y) = 100 + 400y - 40y^2 + 3y^3
\]

a. Write a function representing the profit of the firm as a function of the given output price and output level.

b. What is the profit maximizing level of output for this firm? Verify that the output level you choose is the profit maximizing point.
c. What is the marginal cost of the firm at this output level?

d. What is the variable cost of production at this output level?
Problem 2. Consider a competitive firm with the following production function where \( y \) represents the level of output of the firm and \( x \) represents the level of the single variable input.

\[
y = f(x) = 50x + 60x^2 - 2x^3
\]

The firm faces an output price of \( p = 12 \) and an input price of \( w = 3192 \).

a. Write a function representing the profit of the firm as a function of the given output and input prices and the input level.

b. What is the profit maximizing level of input for this firm? Verify that the input level you choose is the profit maximizing point.
c. What is the level of output of the firm at the profit maximizing input level?

d. What is the marginal product \( MP_x \) of the variable input at the profit maximizing input level?

e. Verify that \( pMP_x = w \) at the profit maximizing input level.
Problem 3. For each of the following problems, write an equation that represents profit as a function of the two inputs $x_1$ and $x_2$. Output price is represented by $p$, the price of the first input by $w_1$ and the price of the second input as $w_2$. Write it in the form $\pi = pf(x_1, x_2) - w_1x_1 - w_2x_2$ and then simplify the expression. Then find critical values of $x_1$ and $x_2$. Verify that profit is maximized at these critical values.

a. 

$$f(x_1, x_2) = 100x_1 + 50x_2 - x_1^2 + x_1x_2 - 3x_2^2$$

$$p = 6$$

$$w_1 = 360, \quad w_2 = 90$$
b. 

\[ f(x_1, x_2) = x_1^{3/5} x_2^{3/10} \]

\[ p = 20 \]

\[ w_1 = 12, \quad w_2 = 2 \]
Problem 4. For each of the following problems you are given a function \( L(x_1, x_2, \lambda) \). In each case find the first partial derivatives of \( L(x_1, x_2, \lambda) \) with respect to \( x_1, x_2, \) and \( \lambda \). Then find the elements of the following matrix

\[
H_B = \begin{bmatrix}
\frac{\partial^2 L(x_1, x_2, \lambda)}{\partial x_1 \partial x_1} & \frac{\partial^2 L(x_1, x_2, \lambda)}{\partial x_1 \partial x_2} & -\frac{\partial^2 L(x_1, x_2, \lambda)}{\partial x_1 \partial \lambda} \\
\frac{\partial^2 L(x_1, x_2, \lambda)}{\partial x_2 \partial x_1} & \frac{\partial^2 L(x_1, x_2, \lambda)}{\partial x_2 \partial x_2} & -\frac{\partial^2 L(x_1, x_2, \lambda)}{\partial x_2 \partial \lambda} \\
\frac{\partial^2 L(x_1, x_2, \lambda)}{\partial \lambda \partial x_1} & \frac{\partial^2 L(x_1, x_2, \lambda)}{\partial \lambda \partial x_2} & -\frac{\partial^2 L(x_1, x_2, \lambda)}{\partial \lambda \partial \lambda}
\end{bmatrix}
\]

a. 
\[
L = 3x_1 + 8x_2 - \lambda(20x_1 + 40x_2 - x_1^2 + x_1 x_2 - x_2^2 - 901)
\]
b. 

\[ \mathcal{L} = 60x_1 + 24x_2 - \lambda(60x_1 + 50x_2 - x_1^2 + x_1x_2 - 2x_2^2 - 1732) \]
c. 

\[ \mathcal{L} = x_1^{1/4} x_2^{1/2} - \lambda (6x_1 + 4x_2 - 108) \]
d. 

\[ \mathcal{L} = 27x_1 + 16x_2 - \lambda \left( x_1^{1/3} x_2^{1/4} - 12 \right) \]
Problem 5. Solve the following system of equations for $x_1, x_2, \lambda$. The easiest way to proceed is to move $-\lambda(\cdots)$ to the right hand side of each of the first two equations, then take the ratio of these two equations, solve for $x_1$ in terms of $x_2$, and then substitute in the third equation.

\begin{align*}
3 - \lambda(20 - 2x_1 + x_2) &= 0 \quad (1a) \\
8 - \lambda(40 + x_1 - 2x_2) &= 0 \quad (1b) \\
901 - 20x_1 + x_1^2 - 40x_1 - x_1x_2 + x_2^2 &= 0 \quad (1c)
\end{align*}
Problem 6. Solve the following system of equations for $x_1$, $x_2$, $\lambda$. The easiest way to proceed is to move $-\lambda(\cdots)$ to the right hand side of each of the first two equations, then take the ratio of these two equations, solve for $x_1$ in terms of $x_2$, and then substitute in the third equation.

\begin{align*}
27 - \frac{\lambda}{3} x_1^{-2/3} x_2^{1/4} &= 0 \quad (2a) \\
16 - \frac{\lambda}{4} x_1^{1/3} x_2^{-3/4} &= 0 \quad (2b) \\
12 - x_1^{1/3} x_2^{1/4} &= 0 \quad (2c)
\end{align*}