For your information, the bordered Hessian in the constrained optimization problem written as

\[ L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda g(x_1, x_2) \]

is given by

\[
H_B = \begin{bmatrix}
\frac{\partial^2 L(x_1, x_2, \lambda)}{\partial x_1 \partial x_1} & \frac{\partial^2 L(x_1, x_2, \lambda)}{\partial x_1 \partial x_2} & \frac{\partial g(x_1, x_2)}{\partial x_1} \\
\frac{\partial^2 L(x_1, x_2, \lambda)}{\partial x_2 \partial x_1} & \frac{\partial^2 L(x_1, x_2, \lambda)}{\partial x_2 \partial x_2} & \frac{\partial g(x_1, x_2)}{\partial x_2} \\
\frac{\partial g(x_1, x_2)}{\partial x_1} & \frac{\partial g(x_1, x_2)}{\partial x_2} & 0
\end{bmatrix}
\]
Problem 1. Consider a firm with a production function given by

\[ y = f(x_1, x_2) = 60x_1 + 50x_2 - x_1^2 + x_1x_2 - 2x_2^2 \]

The firm faces prices and a cost constraint given by

\[ w_1 = 60 \]
\[ w_2 = 24 \]
\[ c_0 = 2376 \]

Find the maximum output the firm can produce given the cost constraint and the stated prices. Verify that this output is a maximum. Some of the optimal values are as follows

\[ x_1 = 32, \quad x_2 = 19 \]

\[ \lambda = \frac{1}{4}, \quad \frac{\partial^2 L(x_1, x_2, \lambda)}{\partial x_1 \partial x_1} = -2 \]

\[ |H_B| = 18432 \]
WORKSPACE
Problem 2. Consider a firm with a production function given by

\[ y = f(x_1, x_2) = 20x_1 + 40x_2 - x_1^2 + x_1x_2 - x_2^2 \]

The firm faces prices and an output constraint given by

\[ w_1 = 3 \]
\[ w_2 = 8 \]
\[ y_0 = 901 \]

Find the minimum cost way to produce the given level of output. Verify that this cost is a minimum. Some of the optimal values are as follows:

\[ x_1 = \frac{94}{3}, \quad x_2 = \frac{119}{3} \]
\[ \lambda = -1, \quad \frac{\partial^2 L(x_1, x_2, \lambda)}{\partial x_1 \partial x_1} = -2 \]
\[ |H_{BE}| = 194 \]
\[ x_1 = 22, \quad x_2 = 27 \]
\[ \lambda = 1, \quad \frac{\partial^2 L(x_1, x_2, \lambda)}{\partial x_1 \partial x_1} = 2 \]
\[ |H_{BE}| = -194 \]
WORKSPACE
Problem 3. Consider a consumer with a utility function given by

\[ u = u(x_1, x_2) = x_1^{1/4}x_2^{1/2} \]

The consumer faces prices and a budget constraint given by

\[ p_1 = 6 \]
\[ p_2 = 4 \]
\[ c_0 = 108 \]

Find the maximum utility the consumer can obtain given the budget constraint and the stated prices. Verify that this utility is a maximum. Some of the optimal values are as follows

\[ x_1 = 6, \quad x_2 = 18 \]

\[ \lambda = \frac{1}{8 \times 2^{1/4} \times 3^{3/4}} \]
\[ \frac{\partial^2 L(x_1, x_2, \lambda)}{\partial x_1 \partial x_1} = -\frac{1}{32} \left( \frac{3}{2} \right)^{1/4} \]

\[ |H_B| = \left( \frac{3}{2} \right)^{1/2} \]
Problem 4. Consider a consumer with a utility function given by

\[ u = u(x_1, x_2) = x_1^{1/3}x_2^{1/4} \]

The consumer faces prices and a utility target given by

\[ p_1 = 27 \]
\[ p_2 = 16 \]
\[ u_0 = 12 \]

Find the minimum cost way to obtain the given level of utility. Verify that this cost is a minimum. Some of the optimal values are as follows

\[ x_1 = 64, \quad x_2 = 81 \]
\[ \lambda = 432, \quad \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1 \partial x_1} = \frac{9}{32} \]
\[ |H_B| = \frac{-7}{5184} \]