ECONOMICS 207
SPRING 2006
LABORATORY EXERCISE 10

Problem 1. For each of the following systems of equations, find the solution vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ by appending the right-hand side vector to the coefficient matrix and performing row reduction.

a

$$\begin{pmatrix} 2 & 3 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 13 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{ccccc} 2 & 3 & 1 & 0 & 13 \\ 4 & -2 & 0 & 1 & 2 \end{array} \right) \rightarrow \left(\begin{array}{ccccc} 2 & 3 & 1 & 0 & 13 \\ 0 & -8 & -2 & 1 & -24 \end{array} \right) \rightarrow \left(\begin{array}{ccccc} 2 & 3 & 1 & 0 & 13 \\ 0 & 1 & \frac{1}{4} & \frac{-1}{8} & 3 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{ccccc} 2 & 0 & \frac{1}{4} & \frac{3}{8} & 4 \\ 0 & 1 & \frac{1}{4} & \frac{-1}{8} & 3 \end{array} \right) \rightarrow \left(\begin{array}{ccccc} 1 & 0 & \frac{1}{8} & \frac{3}{16} & 2 \\ 0 & 1 & \frac{1}{4} & \frac{-1}{8} & 3 \end{array} \right)$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left(\begin{array}{c} 2 \\ 3 \end{array} \right)$$

b

$$\begin{pmatrix} 3 & 1 \\ 6 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccccc} 3 & 1 & 1 & 0 & 5 \\ 6 & -3 & 0 & 1 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccccc} 3 & 1 & 1 & 0 & 5 \\ 0 & -5 & -2 & 1 & -10 \end{array} \right) \rightarrow \left(\begin{array}{ccccc} 3 & 1 & 1 & 0 & 5 \\ 0 & 1 & \frac{2}{5} & \frac{-1}{5} & 2 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{ccccc} 3 & 0 & \frac{3}{5} & \frac{1}{5} & 3 \\ 0 & 1 & \frac{2}{5} & \frac{-1}{5} & 2 \end{array} \right) \rightarrow \left(\begin{array}{ccccc} 1 & 0 & \frac{1}{5} & \frac{1}{15} & 1 \\ 0 & 1 & \frac{2}{5} & \frac{-1}{5} & 2 \end{array} \right)$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left(\begin{array}{c} 1 \\ 2 \end{array} \right)$$

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Problem 2. Consider the following matrices.

$$A = \begin{bmatrix} 4 & 2 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & -1 & 4 \\ 1 & 0 & 2 \\ 4 & -1 & 7 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & -3 & 2 \\ -2 & 5 & -2 \\ 4 & -11 & 7 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 7 \end{bmatrix}$$

Compute the following

(i) a Find the determinant of A.

$$Det(A) = 4 \times 2 - 2 \times 3 = 2$$

b Find the inverse of A using the adjoint method.

$$\begin{aligned} Adj(A) &= \begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix} \\ Inv(A) &= \frac{1}{Det(A)} Adj(A) = \frac{1}{2} \begin{bmatrix} 2 & -2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ \frac{-3}{2} & 2 \end{bmatrix} \end{aligned}$$

c Find the inverse of A using row reduction.

$$\begin{aligned} \begin{pmatrix} 4 & 2 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{pmatrix} &\rightarrow \begin{pmatrix} 1 & 0 & 1 & -1 \\ 3 & 2 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 2 & -3 & 4 \end{pmatrix} \\ &\rightarrow \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & \frac{-3}{2} & 2 \end{pmatrix} \\ &\Rightarrow Inv(A) = \begin{pmatrix} 1 & -1 \\ \frac{-3}{2} & 2 \end{pmatrix} \end{aligned}$$

(ii) a Find the determinant of B.

$$Det(B) = (-1) \times (-3) - 1 \times 2 = 1$$

b Find the inverse of B using the adjoint method.

$$Adj(B) = \begin{bmatrix} -3 & -1 \\ -2 & -1 \end{bmatrix}$$

$$Inv(B) = \frac{1}{Det(B)} Adj(B) = \frac{1}{1} \begin{bmatrix} -3 & -1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ -2 & -1 \end{bmatrix}$$

c Find the inverse of B using row reduction.

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & 1 & -2 & -1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} -1 & 0 & 3 & 1 \\ 0 & 1 & -2 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & -2 & -1 \end{pmatrix}$$

$$\Rightarrow Inv(B) = \begin{pmatrix} -3 & -1 \\ -2 & -1 \end{pmatrix}$$

(iii) a Find the determinant of D.

$$\begin{aligned} Det(D) &= 2\times(-1)^2\begin{vmatrix} 0 & 2 \\ -1 & 7 \end{vmatrix} + (-1)\times(-1)^3\begin{vmatrix} 1 & 2 \\ 4 & 7 \end{vmatrix} + 4\times(-1)^4\begin{vmatrix} 1 & 0 \\ 4 & -1 \end{vmatrix} \\ &= 2\times 2 + (-1) + 4\times(-1) = -1 \end{aligned}$$

b Find the inverse of D using the adjoint method.

$$\begin{aligned} Adj(D) &= \begin{bmatrix} \begin{vmatrix} 0 & 2 \\ -1 & 7 \end{vmatrix} & -\begin{vmatrix} -1 & 4 \\ -1 & 7 \end{vmatrix} & \begin{vmatrix} -1 & 4 \\ 0 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 4 & 7 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 4 & 7 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 \\ 4 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 4 & -1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 2 & 3 & -2 \\ 1 & -2 & 0 \\ -1 & -2 & 1 \end{bmatrix} \\ Inv(D) &= \frac{1}{Det(D)}Adj(D) = \frac{1}{-1}\begin{bmatrix} 2 & 3 & -2 \\ 1 & -2 & 0 \\ -1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -3 & 2 \\ -1 & 2 & 0 \\ 1 & 2 & -1 \end{bmatrix} \end{aligned}$$

c Find the inverse of D using row reduction.

$$\begin{aligned} \begin{pmatrix} 2 & -1 & 4 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 4 & -1 & 7 & 0 & 0 & 1 \end{pmatrix} &\to \begin{pmatrix} 2 & -1 & 4 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 1 & -1 & -2 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & -2 & 1 \end{pmatrix} \\ &\to \begin{pmatrix} 2 & 0 & 4 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 & -1 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 0 & -4 & -6 & 4 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 & -1 \end{pmatrix} \\ &\to \begin{pmatrix} 1 & 0 & 0 & -2 & -3 & 2 \\ 0 & 1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 & -1 \end{pmatrix} \\ &\Rightarrow \quad Inv(D) = \begin{bmatrix} -2 & -3 & 2 \\ -1 & 2 & 0 \\ 1 & 2 & -1 \end{bmatrix} \end{aligned}$$

(iv) a Find the determinant of E.

$$\begin{aligned} Det(E) &= 1\times(-1)^2\begin{vmatrix} 5 & -2 \\ -11 & 7 \end{vmatrix} + (-3)\times(-1)^3\begin{vmatrix} -2 & -2 \\ 4 & 7 \end{vmatrix} + 2\times(-1)^4\begin{vmatrix} -2 & 5 \\ 4 & -11 \end{vmatrix} \\ &= 13 + 3\times(-6) + 2\times 2 = -1 \end{aligned}$$

b Find the inverse of E using the adjoint method.

$$\begin{aligned} Adj(E) &= \begin{bmatrix} \begin{vmatrix} 5 & -2 \\ -11 & 7 \end{vmatrix} & -\begin{vmatrix} -3 & 2 \\ -11 & 7 \end{vmatrix} & \begin{vmatrix} -3 & 2 \\ 5 & -2 \end{vmatrix} \\ -\begin{vmatrix} -2 & -2 \\ 4 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 4 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ -2 & -2 \end{vmatrix} \\ \begin{vmatrix} -2 & 5 \\ 4 & -11 \end{vmatrix} & -\begin{vmatrix} 1 & -3 \\ 4 & -11 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ -2 & 5 \end{vmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 13 & -1 & -4 \\ 6 & -1 & -2 \\ 2 & -1 & -1 \end{bmatrix} \\ Inv(E) &= \frac{1}{Det(E)}Adj(E) = \frac{1}{-1}\begin{bmatrix} 13 & -1 & -4 \\ 6 & -1 & -2 \\ 2 & -1 & -1 \end{bmatrix} = \begin{bmatrix} -13 & 1 & 4 \\ -6 & 1 & 2 \\ -2 & 1 & 1 \end{bmatrix} \end{aligned}$$

c Find the inverse of E using row reduction.

$$\begin{aligned} &\begin{pmatrix} 1 & -3 & 2 & 1 & 0 & 0 \\ -2 & 5 & -2 & 0 & 1 & 0 \\ 4 & -11 & 7 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -3 & 2 & 1 & 0 & 0 \\ 0 & -1 & 2 & 2 & 1 & 0 \\ 0 & 1 & -1 & -4 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -3 & 2 & 1 & 0 & 0 \\ 0 & -1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{pmatrix} \\ &\rightarrow \begin{pmatrix} 1 & -3 & 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 6 & -1 & -2 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & -17 & 3 & 6 \\ 0 & 1 & 0 & -6 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{pmatrix} \\ &\rightarrow \begin{pmatrix} 1 & 0 & 0 & -13 & 1 & 4 \\ 0 & 1 & 0 & -6 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{pmatrix} \Rightarrow Inv(E) = \begin{bmatrix} -13 & 1 & 4 \\ -6 & 1 & 2 \\ -2 & 1 & 1 \end{bmatrix} \end{aligned}$$

Problem 3. For each of the following problems, find the critical points. For each critical point state whether the function is at a relative maximum, relative minimum, or otherwise. Check to see if there are points of inflection **at points other than** critical points.

a $f(x) = 3x^4 - 16x^3 + 18x^2$

Set the first order condition equal to 0 to solve the critical points:

$$\begin{aligned} \frac{\partial f(x)}{\partial x} &= 12x^3 - 48x^2 + 36x = 0 \\ &\Rightarrow 12x(-3+x)(-1+x) = 0 \\ &\Rightarrow x = 0 \text{ or } x = 3 \text{ or } x = 1 \end{aligned}$$

Use the second order condition to check the property of the critical points:

$$\begin{aligned} \frac{\partial^2 f(x)}{\partial x \partial x} &= 36x^2 - 96x + 36 \\ For \quad x &= 0, \quad f''(0) = 36 > 0 \Rightarrow x = 0 \quad \text{is a local minimum.} \\ For \quad x &= 3, \quad f''(3) = 72 > 0 \Rightarrow x = 3 \quad \text{is a local minimum.} \\ For \quad x &= 1, \quad f''(3) = -24 < 0 \Rightarrow x = 1 \quad \text{is a local maximum.} \end{aligned}$$

There are no other points that satisfies the first order condition equal to 0, so there is no inflection point.

b $y = x^3 - 3x^2 + 15$

Set the first order condition equal to 0 to solve the critical points:

$$\begin{aligned} \frac{\partial f(x)}{\partial x} &= 3x^2 - 6x = 0 \\ &\Rightarrow 3x(x-2) = 0 \\ &\Rightarrow x = 0 \text{ or } x = 2 \end{aligned}$$

Use the second order condition to check the property of the critical points:

$$\begin{aligned} \frac{\partial^2 f(x)}{\partial x \partial x} &= 6x^2 - 6 \\ For \quad x &= 0, \quad f''(0) = -6 < 0 \Rightarrow x = 0 \quad \text{is a local maximum.} \\ For \quad x &= 2, \quad f''(2) = 18 > 0 \Rightarrow x = 3 \quad \text{is a local minimum.} \end{aligned}$$

There are no other points that satisfies the first order condition equal to 0, so there is no inflection point.

c $f(x) = \frac{x}{x^2 + 1}$

Set the first order condition equal to 0 to solve the critical points:

$$\begin{aligned} \frac{\partial f(x)}{\partial x} &= \frac{(x^2 + 1) - 2x^2}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2} = \frac{(1-x)(1+x)}{(x^2 + 1)^2} = 0 \\ &\Rightarrow (1-x)(1+x) = 0 \\ &\Rightarrow x = 1 \text{ or } x = -1 \end{aligned}$$

Use the second order condition to check the property of the critical points:

$$\frac{\partial^2 f(x)}{\partial x \partial x} = \frac{(x^2+1)(2x^3-6x)}{(x^2 \ + \ 1)^4}$$

$$For \quad x = 1, \quad f''(1) = \frac{-1}{2} < 0 \Rightarrow x = 1 \quad \text{is a local maximum.}$$

$$For \quad x = -1, \quad f''(2) = \frac{1}{2} > 0 \Rightarrow x = -1 \quad \text{is a local minimum.}$$

There are no other points that satisfies the first order condition equal to 0, so there is no inflection point.