ECONOMICS 207
SPRING 2006
LABORATORY EXERCISE 11

Consider the following matrices and vectors.

$$A = \begin{bmatrix} 3 & 6 \\ 4 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{2} & 1 \\ 2 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 3 & -5 \\ 5 & -1 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & -2 & 2 \\ -6 & 7 & -5 \\ 3 & -2 & \frac{9}{2} \end{bmatrix}$$

$$a = \begin{bmatrix} 12 \\ 19 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \quad c = \begin{bmatrix} 6 \\ -15 \\ 16 \end{bmatrix}, \quad d = \begin{bmatrix} 14 \\ -39 \\ 26 \end{bmatrix}$$

Problem 1. a Find the solution vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ by appending the right-hand side vector to the coefficient matrix and performing row reduction.

$$Ax = a$$

$$\begin{pmatrix} 3 & 6 & 1 & 0 & 12 \\ 4 & 9 & 0 & 1 & 19 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 6 & 1 & 0 & 12 \\ 0 & 1 & -\frac{4}{3} & 1 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 0 & 9 & -6 & -6 \\ 0 & 1 & -\frac{4}{3} & 1 & 3 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 3 & -2 & -2 \\ 0 & 1 & -\frac{4}{3} & 1 & 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

Date: 11 September 2006.

b Find the inverse of the matrix A using the adjoint method.

$$\begin{aligned}
Det(A) &= 3 \times 9 - 4 \times 6 = 27 - 24 = 3 \\
Adj(A) &= \begin{bmatrix} 9 & -6 \\ -4 & 3 \end{bmatrix} \\
Inv(A) &= \frac{1}{Det(A)} Adj(A) = \frac{1}{3} \begin{bmatrix} 9 & -6 \\ -4 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 3 & -2 \\ -\frac{4}{3} & 1 \end{bmatrix}
\end{aligned}$$

c Find the inverse of the matrix A using row reduction method.

$$\begin{aligned}
\begin{pmatrix} 3 & 6 & 1 & 0 \\ 4 & 9 & 0 & 1 \end{pmatrix} &\rightarrow \begin{pmatrix} 3 & 6 & 1 & 0 \\ 0 & 1 & -\frac{4}{3} & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 0 & 9 & -6 \\ 0 & 1 & -\frac{4}{3} & 1 \end{pmatrix} \\
&\rightarrow \begin{pmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -\frac{4}{3} & 1 \end{pmatrix} \\
&\Rightarrow Inv(A) = \begin{bmatrix} 3 & -2 \\ -\frac{4}{3} & 1 \end{bmatrix}
\end{aligned}$$

Problem 2. a Find the solution vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ by appending the right-hand side vector to the coefficient matrix and performing row reduction.

$$Bx = b$$

$$\begin{pmatrix} \frac{1}{2} & 1 & 1 & 0 & 0 \\ 2 & 5 & 0 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} \frac{1}{2} & 1 & 1 & 0 & 0 \\ 0 & 1 & -4 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} \frac{1}{2} & 0 & 5 & -1 & 2 \\ 0 & 1 & -4 & 1 & -2 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 10 & -2 & 4 \\ 0 & 1 & -4 & 1 & -2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

b Find the inverse of the matrix B using the adjoint method.

$$
\begin{aligned}
Det(B) &= \frac{1}{2} \times 5 - 2 \times 1 = \frac{1}{2} \\
Adj(B) &= \begin{bmatrix} 5 & -1 \\ -2 & \frac{1}{2} \end{bmatrix} \\
Inv(B) &= \frac{1}{Det(B)} Adj(B) = \frac{1}{\frac{1}{2}} \begin{bmatrix} 5 & -1 \\ -2 & \frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} 10 & -2 \\ -4 & 1 \end{bmatrix}
\end{aligned}
$$

c Find the inverse of the matrix B using row reduction method.

$$\begin{pmatrix} \frac{1}{2} & 1 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} \frac{1}{2} & 1 & 1 & 0 \\ 0 & 1 & -4 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} \frac{1}{2} & 0 & 5 & -1 \\ 0 & 1 & -4 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 10 & -2 \\ 0 & 1 & -4 & 1 \end{pmatrix}$$

$$\Rightarrow Inv(B) = \begin{bmatrix} 10 & -2 \\ -4 & 1 \end{bmatrix}$$

d Find the solution vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ to Bx=b by using the inverse you found in part c

$$\begin{aligned} Bx &= b \\ &\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = B^{-1} \times b = \begin{bmatrix} 10 & -2 \\ -4 & 1 \end{bmatrix} \times \begin{pmatrix} 0 \\ -2 \end{pmatrix} \\ &\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \end{aligned}$$

Problem 3. a Find the solution vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ by appending the right-hand side vector to the coefficient matrix and performing row reduction.

$$Cx = c$$

$$\begin{pmatrix} 1 & -1 & 2 & 1 & 0 & 0 & 6 \\ -2 & 3 & -5 & 0 & 1 & 0 & -15 \\ 5 & -1 & 5 & 0 & 0 & 1 & 16 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 & 1 & 0 & 0 & 6 \\ 0 & 1 & -1 & 2 & 1 & 0 & -3 \\ 0 & 4 & -5 & -5 & 0 & 1 & -14 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & -1 & 2 & 1 & 0 & 0 & 6 \\ 0 & 1 & -1 & 2 & 1 & 0 & -3 \\ 0 & 0 & -1 & -13 & -4 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 & 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 15 & 5 & -1 & -1 \\ 0 & 0 & -1 & -13 & -4 & 1 & -2 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & -1 & 2 & 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 15 & 5 & -1 & -1 \\ 0 & 0 & 1 & 13 & 4 & -1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 16 & 5 & -1 & 5 \\ 0 & 1 & 0 & 15 & 5 & -1 & -1 \\ 0 & 0 & 1 & 13 & 4 & -1 & 2 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & -10 & -3 & 1 & 1 \\ 0 & 1 & 0 & 15 & 5 & -1 & -1 \\ 0 & 0 & 1 & 13 & 4 & -1 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

b Find the inverse of the matrix C using row reduction method.

$$\begin{pmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ -2 & 3 & -5 & 0 & 1 & 0 \\ 5 & -1 & 5 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 4 & -5 & -5 & 0 & 1 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & -1 & -13 & -4 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 15 & 5 & -1 \\ 0 & 0 & -1 & -13 & -4 & 1 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 15 & 5 & -1 \\ 0 & 0 & 1 & 13 & 4 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 16 & 5 & -1 \\ 0 & 1 & 0 & 15 & 5 & -1 \\ 0 & 0 & 1 & 13 & 4 & -1 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & -10 & -3 & 1 \\ 0 & 1 & 0 & 15 & 5 & -1 \\ 0 & 0 & 1 & 13 & 4 & -1 \end{pmatrix} \Rightarrow Inv(C) = \begin{pmatrix} -10 & -3 & 1 \\ 15 & 5 & -1 \\ 13 & 4 & -1 \end{pmatrix}$$

Problem 4. a Find the solution vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ by appending the right-hand side vector to the coefficient matrix and performing row reduction.

$$Dx = d$$

$$\begin{pmatrix} 2 & -2 & 2 & 1 & 0 & 0 & 14 \\ -6 & 7 & -5 & 0 & 1 & 0 & -39 \\ 3 & -2 & \frac{9}{2} & 0 & 0 & 1 & 26 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -2 & 2 & 1 & 0 & 0 & 14 \\ 0 & 1 & 1 & 3 & 1 & 0 & 3 \\ 0 & 1 & \frac{3}{2} & -\frac{3}{2} & 0 & 1 & 5 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 2 & -2 & 2 & 1 & 0 & 0 & 14 \\ 0 & 1 & 1 & 3 & 1 & 0 & 3 \\ 0 & 0 & \frac{1}{2} & -\frac{9}{2} & -1 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -2 & 2 & 1 & 0 & 0 & 14 \\ 0 & 1 & 1 & 3 & 1 & 0 & 3 \\ 0 & 0 & 1 & -9 & -2 & 2 & 4 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 2 & -2 & 2 & 1 & 0 & 0 & 14 \\ 0 & 1 & 0 & 12 & 3 & -2 & -1 \\ 0 & 0 & 1 & -9 & -2 & 2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 0 & 2 & 25 & 6 & -4 & 12 \\ 0 & 1 & 0 & 12 & 3 & -2 & -1 \\ 0 & 0 & 1 & -9 & -2 & 2 & 4 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 2 & 0 & 0 & 43 & 10 & -8 & 4 \\ 0 & 1 & 0 & 12 & 3 & -2 & -1 \\ 0 & 0 & 1 & -9 & -2 & 2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & \frac{43}{2} & 5 & -4 & 2 \\ 0 & 1 & 0 & 12 & 3 & -2 & -1 \\ 0 & 0 & 1 & -9 & -2 & 2 & 4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$$

b Find the inverse of the matrix D using the adjoint method.

$$Adj(D) = \begin{bmatrix} \begin{vmatrix} 7 & -5 \\ -2 & \frac{9}{2} \end{vmatrix} & -\begin{vmatrix} -2 & 2 \\ -2 & \frac{9}{2} \end{vmatrix} & \begin{vmatrix} -2 & 2 \\ 7 & -5 \end{vmatrix} \\ -\begin{vmatrix} -6 & -5 \\ 3 & \frac{9}{2} \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 3 & \frac{9}{2} \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ -6 & -5 \end{vmatrix} \\ \begin{vmatrix} -6 & 7 \\ 3 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & -2 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} 2 & -2 \\ -6 & 7 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} \frac{43}{2} & 5 & -4 \\ 12 & 3 & -2 \\ -9 & -2 & 2 \end{bmatrix}$$

$$Det(D) = 2 \times (-1)^2 \times \begin{vmatrix} 7 & -5 \\ -2 & \frac{9}{2} \end{vmatrix} - 2 \times (-1)^3 \times \begin{vmatrix} -6 & -5 \\ 3 & \frac{9}{2} \end{vmatrix} + 2 \times (-1)^4 \times \begin{vmatrix} -6 & 7 \\ 3 & -2 \end{vmatrix}$$

$$= 2 \times \frac{43}{2} + 2 \times (-12) + 2 \times (-9) = 43 - 24 - 18 = 1$$

$$Inv(D) = \frac{1}{Det(D)} Adj(D) = \frac{1}{1} \begin{bmatrix} \frac{43}{2} & 5 & -4 \\ 12 & 3 & -2 \\ -9 & -2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{43}{2} & 5 & -4 \\ 12 & 3 & -2 \\ -9 & -2 & 2 \end{bmatrix}$$

c Find the solution vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ to the equation Dx=d by using the inverse you found in part b

$$\begin{aligned} Dx &= d \\ \Rightarrow & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = Inv(D) \times d = \begin{bmatrix} \frac{43}{2} & 5 & -4 \\ 12 & 3 & -2 \\ -9 & -2 & 2 \end{bmatrix} \times \begin{pmatrix} 14 \\ -39 \\ 26 \end{pmatrix} \\ \Rightarrow & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \end{aligned}$$

Problem 5. For each of the following problems, find the critical points. For each critical point state whether the function is at a relative maximum, relative minimum, or otherwise. Check to see if there are points of inflection **at points other than** critical points.

a $f(x) = \frac{3}{5}x^5 - \frac{11}{2}x^4 - \frac{16}{3}x^3$

Set the first order condition equal to 0 to solve the critical points:

$$\begin{aligned}\frac{\partial f(x)}{\partial x} &= 3x^4 - 22x^3 - 16x^2 = x^2(-8+x)(2+3x) = 0 \\ &\Rightarrow \quad x = 0 \quad or \quad x = 8 \quad or \quad x = -\frac{2}{3}\end{aligned}$$

Use the second order condition to check the property of the critical points:

$$\begin{aligned}\frac{\partial^2 f(x)}{\partial x^2} &= 12x^3 - 66x^2 - 32x \\ \frac{\partial^3 f(x)}{\partial x^3} &= 36x^2 - 132x - 32 \\ For \quad x &= 0, \qquad f''(0) = 0. \text{ (n is even.)}, \qquad f'''(0) = -32 < 0 \quad \text{(n is odd)}. \\ &\Rightarrow \quad x = 0 \; is \; neither \; a \; local \; maximum \; nor \; a \; local \; minimum. \\ For \quad x &= 8, \quad f''(8) = 12 \times 8^3 - 66 \times 8^2 - 32 \times 8 = 1664 > 0 \\ &\Rightarrow \quad x = 8 \; is \; a \; local \; minimum. \\ For \quad x &= -\frac{2}{3}, \quad f''(8) = 12 \times (-\frac{2}{3})^3 - 66 \times (-\frac{2}{3})^2 - 32 \times (-\frac{2}{3}) = -\frac{104}{9} < 0 \\ &\Rightarrow \quad x = -\frac{2}{3} \; is \; a \; local \; maximum.\end{aligned}$$

b $f(x) = \dfrac{x^4}{4} - \dfrac{x^3}{6} - 2x^2 + 2x,$ One of the roots is x=2.

Set the first order condition equal to 0 to solve the critical points:

$$\begin{aligned}\frac{\partial f(x)}{\partial x} &= x^3 - \frac{1}{2}x^2 - 4x + 2 = \frac{1}{2}(x-2)(x+2)(2x-1) = 0 \\ &\Rightarrow \quad x = 2 \quad or \quad x = -2 \quad or \quad x = \frac{1}{2}\end{aligned}$$

Use the second order condition to check the property of the critical points:

$$\begin{aligned}\frac{\partial^2 f(x)}{\partial x^2} &= 3x^2 - x - 4 \\ For \quad x &= 2, \qquad f''(0) = 6 > 0. \\ &\Rightarrow x = 2 \; is \; a \; local \; minimum. \\ For \quad x &= -2, \qquad f''(0) = 10 > 0. \\ &\Rightarrow x = -2 \; is \; a \; local \; minimum. \\ For \quad x &= \frac{1}{2}, \qquad f''(0) = -\frac{15}{4} < 0. \\ &\Rightarrow x = \frac{1}{2} \; is \; a \; local \; maximum.\end{aligned}$$

Problem 6. Find the definite integral for each of the following functions.

a $\int_5^8 \left(4x^2 - 30x + 100\right) dx$

$$\begin{aligned}\int_5^8 \left(4x^2 - 30x + 100\right) dx &= (\frac{4}{3}x^3 - 15x^2 + 100x)|_5^8 \\ &= (\frac{4}{3} \times 8^3 - 15 \times 8^2 + 100 \times 8) - (\frac{4}{3} \times 5^3 - 15 \times 5^2 + 100 \times 5) \\ &= \frac{1568}{3} - \frac{875}{3} = 231\end{aligned}$$

b $\int_3^9 \left(2x^3 + 4x^2 + 30x + 5\right) dx$

$$\begin{aligned}&= (\frac{1}{2}x^4 + \frac{4}{3}x^3 + 15x^2 + 5x)|_3^9 \\ &= (\frac{1}{2} \times 9^4 + \frac{4}{3} \times 9^3 + 15 \times 9^2 + 5 \times 9) - (\frac{1}{2} \times 3^4 + \frac{4}{3} \times 3^3 + 15 \times 3^2 + 5 \times 3) \\ &= \frac{11025}{2} - \frac{453}{2} = 5286\end{aligned}$$

Problem 7. Find all first partial derivatives of each of the following

a $y = 3\,x_1^3 - 9\,x_2^2 + 50\,x_3 + 20$

$$\begin{aligned}\frac{\partial y}{\partial x_1} &= 9x_1^2\\ \frac{\partial y}{\partial x_2} &= -18x_2\\ \frac{\partial y}{\partial x_3} &= 50\end{aligned}$$

b $y = 3\,x_1^3\,x_2^2\,x_3$

$$\begin{aligned}\frac{\partial y}{\partial x_1} &= 9\,x_1^2\,x_2^2\,x_3\\ \frac{\partial y}{\partial x_2} &= 6\,x_1^3\,x_2\,x_3\\ \frac{\partial y}{\partial x_3} &= 3\,x_1^3\,x_2^2\end{aligned}$$

c $y = 3\,x_1^3\,x_2^2 + 2\,x_3\,x_1^2$

$$\begin{aligned}\frac{\partial y}{\partial x_1} &= 9\,x_1^2\,x_2^2 + 4\,x_3\,x_1\\ \frac{\partial y}{\partial x_2} &= 6\,x_1^3\,x_2\\ \frac{\partial y}{\partial x_3} &= 2\,x_1^2\end{aligned}$$

d $y = \frac{3\,x_1^3\,x_2^2}{x_1^2\,\ln(3x_2)}$

$$\begin{aligned}y &= \frac{3\,x_1^3\,x_2^2}{x_1^2\,\ln(3x_2)} = \frac{3\,x_1\,x_2^2}{\ln(3x_2)}\\ \frac{\partial y}{\partial x_1} &= \frac{3\,x_2^2}{\ln(3x_2)}\\ \frac{\partial y}{\partial x_2} &= 3x_1\frac{2x_2\ln(3x_2) - x_2}{(\ln(3x_2))^2}\end{aligned}$$

e $y = 3\,x_1^3\,x_2^{0.2}\,x_3^{0.5} - 4\,x_1 - 5x_2 - x_3$

$$\begin{aligned}\frac{\partial y}{\partial x_1} &= 9\,x_1^2\,x_2^{0.2}\,x_3^{0.5} - 4\\ \frac{\partial y}{\partial x_2} &= 0.6\,x_1^3\,x_2^{-0.8}\,x_3^{0.5} - 5\\ \frac{\partial y}{\partial x_3} &= 1.5x_1^3\,x_2^{0.2}\,x_3^{-0.5} - 1\end{aligned}$$

f $y = \frac{3\,x_1^2\,x_3^2}{4\,x_1^2\;e^{x_2^2}}$

$$\begin{aligned} y &= \frac{3\,x_1^2\,x_3^2}{4\,x_1^2\;e^{x_2^2}} = \frac{3\,x_3^2}{4e^{x_2^2}} \\ \frac{\partial y}{\partial x_1} &= 0 \\ \frac{\partial y}{\partial x_2} &= \frac{3\,x_3^2}{4}(e^{-x_2^2})(-2x_2) = \frac{-3x_2\,x_3^2}{2}(e^{-x_2^2}) \\ \frac{\partial y}{\partial x_3} &= \frac{3}{4e^{x_2^2}} \times (2x_3) = \frac{3\,x_3}{2e^{x_2^2}} \end{aligned}$$