ECONOMICS 207
SPRING 2006
LABORATORY EXERCISE 5 KEY

Problem 1. Solve the following equations for x.

a $\frac{5x - 2}{3x+8} = \frac{9}{10}$

$$\begin{aligned} \frac{5x - 2}{3x+8} &= \frac{9}{10} \Rightarrow 10(5x - 2) = 9(3x+8),\ x \neq \frac{-8}{3} \\ &\Rightarrow 50x - 20 = 27x + 72 \\ &\Rightarrow 23x = 92 \\ &\Rightarrow x = 4 \end{aligned}$$

b $8x^2 - 22x + 15 = 0$

$$\begin{aligned} 8x^2 - 22x + 15 &= 0 \\ &\Rightarrow (2x-3)(4x-5) = 0 \\ &\Rightarrow x = \frac{3}{2},\ or\ x = \frac{5}{4} \end{aligned}$$

c $20x^2 + 14x - 24 = 0$

$$\begin{aligned} 20x^2 + 14x - 24 &= 0 \\ &\Rightarrow (4x+6)(5x-4) = 0 \\ &\Rightarrow x = -\frac{3}{2},\ or\ x = \frac{4}{5} \end{aligned}$$

Date: 28 December 2006.

d $729x_1^{-5/4} - 343x_1^{-1/2} = 0, \quad (x_1 = \frac{6561}{2401})$

$$\begin{aligned}
729x_1^{-5/4} - 343x_1^{-1/2} &= 0 \\
&\Rightarrow 729x_1^{-5/4} = 343x_1^{-1/2} \\
&\Rightarrow \frac{x_1^{-1/2}}{x_1^{-5/4}} = \frac{729}{343} \\
&\Rightarrow x_1^{3/4} = \frac{729}{343} = (\frac{9}{7})^3 \\
&\Rightarrow x_1 = \left(\frac{9}{7}\right)^{3\times\frac{4}{3}} = \left(\frac{9}{7}\right)^4 = \frac{6561}{2401}
\end{aligned}$$

Problem 2. Solve the following systems of equations for x_1 and x_2 using the method of substitution.

a

$\{x_1 = 81, x_2 = 16\}$

$$144x_1^{-1/2}x_2^{1/4} - 32 = 0$$

$$72x_1^{1/2}x_2^{-3/4} - 81 = 0$$

Take the ratio of these two equations, we have,

$$\frac{144x_1^{-1/2}x_2^{1/4}}{72x_1^{1/2}x_2^{-3/4}} = \frac{32}{81} \Rightarrow \frac{2x_2}{x_1} = \frac{32}{81} \Rightarrow x_2 = \frac{16}{81}x_1$$

$$Substitute\ x_2 = \frac{16}{81}x_1 \text{ into the first equation, we obtain,}$$

$$144x_1^{-1/2}\left(\frac{16}{81}x_1\right)^{1/4} = 32$$

$$\Rightarrow 144x_1^{-1/2}(\frac{2^4}{3^4}x_1)^{1/4} = 32$$

$$\Rightarrow 144\left(\frac{2}{3}\right)x_1^{-1/4} = 32$$

$$\Rightarrow x_1^{-1/4} = \frac{32 \times 3}{144 \times 2} = \frac{1}{3}$$

$$\Rightarrow x_1 = \left(\frac{1}{3}\right)^{-4} = 3^4 = 81$$

$$x_2 = \frac{64}{81}x_1 = \frac{64}{81} \times 81 = 64$$

So, $x_1 = 81, x_2 = 64.$

b

$\{x_1 \ = \ 9, \ x_2 \ = \ 1\}$

$$15x_1^{-1/2}x_2^{2/5} \ - \ 5 \ = 0$$
$$12x_1^{1/2}x_2^{-3/5} \ - \ 36 \ = 0$$

Take the ratio of these two equations, we have,

$$\frac{15x_1^{-1/2}x_2^{2/5}}{12x_1^{1/2}x_2^{-3/5}} = \frac{5}{36} \Rightarrow \frac{15x_2}{12x_1} = \frac{5}{36} \Rightarrow x_2 = \frac{1}{9}x_1$$

$Substitute\ x_2 = \frac{1}{9}x_1$ into the first equation, we obtain,

$$15x_1^{-1/2}\left(\frac{1}{9}x_1\right)^{2/5} = 5$$
$$\Rightarrow x_1^{-1/2}(\frac{1}{9}x_1)^{2/5} = \frac{1}{3}$$
$$\Rightarrow \left(\frac{1}{9}\right)^{2/5} x_1^{-1/10} = \frac{1}{3}$$
$$\Rightarrow x_1^{-1/10} = \frac{1}{3} \times (\frac{1}{3})^{-4/5} = (\frac{1}{3})^{1/5}$$
$$\Rightarrow x_1 = \left(\frac{1}{3}\right)^{(1/5)\times -10} = (\frac{1}{3})^{-2} = 9$$
$$x_2 = \frac{1}{9}x_1 = \frac{1}{9} \times 9 = 1$$

So, $x_1 = 9, x_2 = 1.$

Problem 3. Solve the following systems of equations for x_1 and x_2 first using the method of substitution and then using the method of elimination.

a

$\{x_1 = -2, x_2 = -3\}$

$$2x_1 - 3x_2 = 5$$
$$4x_1 + 3x_2 = -17$$

Method of substitution :

By the first equation, we can get,

$$x_1 = (5 + 3x_2)/2$$

Substitute $x_1 = \frac{(5+3x_2)}{2}$ into the second equation, we obtain,

$$4 \times \frac{(5+3x_2)}{2} + 3x_2 = -17$$
$$\Rightarrow 10 + 6x_2 + 3x_2 = -17$$
$$\Rightarrow 9x_2 = -27$$
$$\Rightarrow x_2 = -3$$
$$x_1 = \frac{(5+3x_2)}{2} = \frac{(5-9)}{2} = -2$$

So, the solution is $x_1 = -2, \ x_2 = -3.$

Method of elimination :

Multiply the first equation by -2 and add it to the second equation. This gives,

$$-2(2x_1 - 3x_2) = -2 \times 5 \Rightarrow -4x_1 + 6x_2 = -10$$
$$4x_1 + 3x_2 = -17$$
$$\Rightarrow -4x_1 + 6x_2 + 4x_1 + 3x_2 = -27$$
$$\Rightarrow 9x_2 = -27 \Rightarrow x_2 = -3$$

Multiply the above equation by 3 and add it to the first equation. This gives,

$$3x_2 = -9$$
$$2x_1 - 3x_2 = 5$$
$$\Rightarrow 2x_1 = -4 \Rightarrow x_1 = -2.$$

So, the solution is $x_1 = -2, \ x_2 = -3.$

b

$\{x_1 = 2, x_2 = 4\}$

$$-2x_1 + 4x_2 = 12$$
$$3x_1 + 6x_2 = 30$$

Method of substitution :

By the first equation, we can get,

$$x_1 = \frac{(12 - 4x_2)}{-2} = 2x_2 - 6$$

Substitute $x_1 = 2x_2 - 6$ into the second equation, we obtain,

$$3 \times (2x_2 - 6) + 6x_2 = 30$$
$$\Rightarrow 6x_2 - 18 + 6x_2 = 30$$
$$\Rightarrow 12x_2 = 48 \Rightarrow x_2 = 4$$
$$x_1 = 2x_2 - 6 = 2$$

So, the solution is $x_1 = 2,\ x_2 = 4.$

Method of elimination :

Multiply the first equation by $\frac{3}{2}$ and add it to the second equation.

$$-3x_1 + 6x_2 = 18$$
$$3x_1 + 6x_2 = 30$$
$$\Rightarrow 12x_2 = 48 \Rightarrow x_2 = 4.$$

Multiply the above equation by -6 and add it to the second equation.

$$-6x_2 = -24$$
$$3x_1 + 6x_2 = 30$$
$$\Rightarrow 3x_1 = 6 \Rightarrow x_1 = 2$$

So, the solution is $x_1 = 2,\ x_2 = 4.$

Problem 4. Solve the following system of equations for x_1, x_2, and x_3 first using the method of substitution and then using the method of elimination.
$\{x_1 = 2,\ x_2 = -1,\ x_3 = 3\}$

$$\begin{aligned} x_1 + 2x_2 + 6x_3 &= 18 \\ -2x_1 - 3x_2 - 10x_3 &= -31 \\ 2x_1 + 3x_2 + 11x_3 &= 34 \end{aligned}$$

By the first equation we have, $x_1 = 18 - 2x_2 - 6x_3$

Substitute the above equation into the second equation, we have,

$$\begin{aligned} -2(18 - 2x_2 - 6x_3) - 3x_2 - 10x_3 &= -31 \\ &\Rightarrow x_2 + 2x_3 = 5 \Rightarrow x_2 = 5 - 2x_3 \\ &\Rightarrow x_1 = 18 - 2(5 - 2x_3) - 6x_3 = 8 - 2x_3 \end{aligned}$$

Substitute the above two equations into the third equation, we have,

$$\begin{aligned} 2(8 - 2x_3) + 3(5 - 2x_3) + 11x_3 &= 34 \\ &\Rightarrow 16 - 4x_3 + 15 - 6x_3 + 11x_3 = 34 \\ &\Rightarrow x_3 = 3 \\ x_1 &= 8 - 2x_3 = 8 - 6 = 2 \\ x_2 &= 5 - 2x_3 = 5 - 6 = -1 \end{aligned}$$

So, the solution is $x_1 = 2,\ x_2 = -1,\ x_3 = 3$

Method of elimination:

Multiply the first equation by 2 and add it to the second equation. This gives,

$$\begin{aligned} 2(x_1 + 2x_2 + 6x_3) &= 2 \times 18 \Rightarrow 2x_1 + 4x_2 + 12x_3 = 36 \\ -2x_1 - 3x_2 - 10x_3 &= -31 \\ &\Rightarrow x_2 + 2x_3 = 5 \end{aligned}$$

Second equation plus the third equation, we get,

$$\begin{aligned} -2x_1 - 3x_2 - 10x_3 &= -31 \\ 2x_1 + 3x_2 + 11x_3 &= 34 \\ &\Rightarrow x_3 = 3 \end{aligned}$$

Multiply the above equation by -2 and add it to the equation $x_2 + 2x_3 = 5$. This gives,

$$\begin{aligned} -2x_3 &= -6 \\ x_2 + 2x_3 &= 5 \\ &\Rightarrow x_2 = -1 \end{aligned}$$

Multiply the above equation by -2, multiply the equation $x_3 = 3$ by -6 and add it to the first equation, we obtain

$$\begin{aligned} -2x_2 &= 2 \\ -6x_3 &= -18 \\ x_1 + 2x_2 + 6x_3 &= 18 \\ &\Rightarrow x_1 = 2 \end{aligned}$$

So, the solution is $x_1 = 2,\ x_2 = -1,\ x_3 = 3$

Problem 5. Find the derivatives of each of the following functions with respect to x.

a $y = 2x^2 + 4x^3$

$$\frac{d\,y}{dx} = 4x + 12x^2$$

b $f(x) = 2x^3 + 4\,e^x$

$$f^{'}(x) = 6x^2 + 4\,e^x$$

c $f(x) = 3x^2 - 4\log[x]$

$$f^{'}(x) = 6x - \frac{4}{x}$$

d $f(x) = -3x^2 + 12x - 4^x$

$$f^{'}(x) = -6x + 12 - \log[4] \times 4^x$$

e $f(x) = 3x^{1/2} + 12x^{1/3} - 4x^{-2}$

$$f^{'}(x) = \frac{3}{2}x^{-1/2} + 4x^{-2/3} + 8x^{-3}$$

f $f(x) = 4x^{-3} - 2x\,e^x$

$$f^{'}(x) = -12x^{-4} - 2\,e^x - 2x\,e^x$$

g $f(x) \ = \ 9x^{1/3} \ + \ 2x^2 \log[x]$

$$\begin{aligned} f^{'}(x) &= 3x^{-2/3} + 4x\log[x] + \frac{2x^2}{x} \\ &= 3x^{-2/3} + 4x\log[x] + 2x \end{aligned}$$

h $f(x) \ = \ (2x \ + \ 5)^3$ Find in two different ways.

$$\begin{aligned} f(x) &= (2x \ + \ 5)^3 = 8x^3 + 60x^2 + 150x + 125 \\ f^{'}(x) &= 24x^2 + 120x + 150 \\ Also, \ f^{'}(x) &= 3(2x \ + \ 5)^2 \times 2 = 24x^2 + 120x + 150 \end{aligned}$$

i $f(x) \ = \ \frac{4x^2}{x^2+2x}$

$$\begin{aligned} f(x) &= \frac{4x^2}{x^2+2x} = \frac{4x}{x+2} \\ f^{'}(x) &= \frac{4(x+2)-4x}{(x+2)^2} = \frac{8}{(x+2)^2} \end{aligned}$$

j $f(x) \ = \ \frac{3x^2+2x}{4x^2+5}$

$$f^{'}(x) = \frac{(6x+2)(4x^2+5)-(8x)(3x^2+2x)}{(4x^2+5)^2} = \frac{-8x^2+30x+10}{(4x^2+5)^2}$$

k $f(x) \ = \ -2x^4 \ + \ 2e^{2x}$

$$f^{'}(x) = -8x^3 + 2e^{2x} \times 2 = -8x^3 + 4e^{2x}$$

l $f(x) \ = \ \frac{3x^2\, e^x}{x^2+3}$

$$f^{'}(x) = \frac{(6xe^x + 3x^2\, e^x)(x^2+3) - 2x(3x^2\, e^x)}{(x^2+3)^2} = \frac{e^x(18x + 3x^4 + 9x^2)}{(x^2+3)^2}$$

Problem 6. Find the derivatives of each of the following functions with respect to x.

a $f(x) \ = \ (x^2 \ + \ 2x)^2$

$$f^{'}(x) = 2(x^2+2x)(2x+2) = 4x^3 + 12x^2 + 8x$$

b $f(x) \ = \ (5x \ - \ 2)(3x \ + \ 4)$ Show two ways.

$$\begin{aligned} f(x) &= (5x \ - \ 2)(3x \ + \ 4) = 15x^2 + 14x - 8 \\ f^{'}(x) &= 30x + 14 \\ Also,\ f^{'}(x) &= 5(3x \ + \ 4) + 3(5x \ - \ 2) = 30x + 14 \end{aligned}$$

c $f(x) \ = \ 4x\, e^{x^2+2x}$

$$f^{'}(x) = 4e^{x^2+2x} + e^{x^2+2x}(2x+2) \times 4x = (4 + 8x^2 + 8x)e^{x^2+2x}$$

d $f(x) \ = \ 12^x\, e^{2x^2+3x}$

$$f^{'}(x) = \ln^{12} \times 12^x \ \times e^{2x^2+3x} + e^{2x^2+3x}(4x+3)12^x = 12^x\, e^{2x^2+3x}(\ln^{12} + 4x + 3)$$

e $f(x) = x^2 \, e^{2x^2 - 5x}$

$$f^{'}(x) = 2xe^{2x^2-5x} + e^{2x^2-5x}(4x-5)x^2 = e^{2x^2-5x}(4x^3 - 5x^2 + 2x)$$

f $f(x) = \log[(x^3 - 4x)^2]$

$$f^{'}(x) = \frac{1}{(x^3-4x)^2} \times 2(x^3 - 4x) \times (3x^2 - 4) = \frac{6x^2 - 8}{(x^3 - 4x)}$$

g $f(x) = \frac{3x \, e^{2x}}{4x^2+2}$

$$f^{'}(x) = \frac{(3\,e^{2x} + 6x\,e^{2x})(4x^2+2) - (8x)(3x\,e^{2x})}{(4x^2+2)^2} = \frac{e^{2x}(24x^3 - 12x^2 + 12x + 6)}{(4x^2+2)^2}$$

h $f(x) = \frac{3x \log[2x^2]}{x^2 + 2x}$

$$\begin{aligned} f^{'}(x) &= \frac{(3\log[2x^2] + 3x \times \frac{4x}{2x^2})(x^2 + 2x) - (2x+2)(3x\log[2x^2])}{(x^2 + 2x)^2} \\ &= \frac{(3\log[2x^2] + 6)(x^2 + 2x) - (2x+2)(3x\log[2x^2])}{(x^2 + 2x)^2} \end{aligned}$$

Problem 7. For each of the following, take the derivative with respect to x_1, set the derivative equal to zero and solve the resulting equation for x_1.

a $\{x_1 = 256\}$
$f(x) = 256x_1^{3/8} - 3x_1$

$$\begin{aligned}
\frac{df(x)}{dx_1} &= 256 \times \frac{3}{8} x_1^{-5/8} - 3 = 0 \\
&\Rightarrow \quad x_1^{-5/8} = \frac{3}{256 \times \frac{3}{8}} = \frac{8}{256} = \frac{2^3}{2^8} = 2^{-5} \\
&\Rightarrow \quad x_1 = 2^{-5 \times \frac{8}{-5}} = 2^8 = 256
\end{aligned}$$

b $\{x_1 = 16\}$

$f(x) = 128 x_1^{1/4} - 4x_1$

$$\begin{aligned}
\frac{df(x)}{dx_1} &= 128 \times \frac{1}{4} x_1^{-3/4} - 4 = 0 \\
&\Rightarrow \quad x_1^{-3/4} = \frac{4}{128 \times \frac{1}{4}} = \frac{1}{8} = 2^{-3} \\
&\Rightarrow \quad x_1 = 2^{-3 \times \frac{4}{-3}} = 2^4 = 16
\end{aligned}$$

c $f(x) = 32 p x_1^{1/4} - 4x_1$

$$\begin{aligned}
\frac{df(x)}{dx_1} &= 8 p x_1^{-3/4} - 4 = 0 \\
&\Rightarrow \quad x_1^{-3/4} = \frac{4}{8p} = (2p)^{-1} \\
&\Rightarrow \quad x_1 = (2p)^{-1 \times \frac{4}{-3}} = (2p)^{\frac{4}{3}}
\end{aligned}$$

d $f(x) = 30 x_1^{3/5} x_2^{1/5} - 6x_1 - 3x_2$

$$\begin{aligned}
\frac{\partial f(x)}{\partial x_1} &= 18 x_1^{-2/5} x_2^{1/5} - 6 = 0 \\
&\Rightarrow \quad x_1^{-2/5} = \frac{1}{3} x_2^{-1/5} \\
&\Rightarrow \quad x_1 = \left(\frac{1}{3} x_2^{-1/5} \right)^{\frac{5}{-2}} = 3^{\frac{5}{2}} x_2^{1/2}
\end{aligned}$$