ECONOMICS 207
SPRING 2006
LABORATORY EXERCISE 6-KEY

Problem 1. Find the derivatives of each of the following functions with respect to x.

a $y = 12x^{1/3} + 4xe^x$

$$\begin{aligned} \frac{dy}{dx} &= 12(\frac{1}{3})x^{-2/3} + 4xe^x + 4e^x \\ &= 4x^{-2/3} + 4e^x(x+1) \end{aligned}$$

b $y = \frac{3x^2 + 2x}{(x^3 + 2)^2}$, Simplify a little.

$$\frac{dy}{dx} = \frac{(6x+2)(x^3 + 2)^2 - 2(x^3 + 2)3x^2(3x^2 + 2x)}{(x^3 + 2)^4}$$

c $y = ln[(x^2 - 2x)^3]$, Simplify a little.

$$\begin{aligned} y &= ln[(x^2 - 2x)^3] = 3\,ln[(x^2 - 2x)] \\ \frac{dy}{dx} &= 3\frac{1}{x^2 - 2x}(2x-2) = 6\frac{x-1}{x^2 - 2x} \end{aligned}$$

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d $y = \frac{3x^2e^{2x^3}}{2x^2 + 4x}$

$$y = \frac{3x^2e^{2x^3}}{2x^2 + 4x} = \frac{3xe^{2x^3}}{2x + 4}$$

$$\frac{dy}{dx} = \frac{(3e^{2x^3} + e^{2x^3}6x^2 3x)(2x + 4) - 2 \times 3xe^{2x^3}}{(2x + 4)^2}$$

$$= \frac{(3e^{2x^3} + e^{2x^3}6x^2 3x)(2x + 4) - 6xe^{2x^3}}{(2x + 4)^2}$$

e Find the derivative with respect to x_1. $y = 1620x_1^{1/5}x_2^{2/5} - 16x_1 - 243x_2$

$$\frac{\partial y}{\partial x_1} = 1620(\frac{1}{5})x_1^{-4/5}x_2^{2/5} - 16$$

$$= 324x_1^{-4/5}x_2^{2/5} - 16$$

f Find the derivative with respect to x_2. $y = 1620x_1^{1/5}x_2^{2/5} - 16x_1 - 243x_2$

$$\frac{\partial y}{\partial x_2} = 1620(\frac{2}{5})x_1^{1/5}x_2^{-3/5} - 243$$

$$= 648x_1^{1/5}x_2^{-3/5} - 243$$

Problem 2. Find the second derivative of each of the following functions with respect to x

a $y = 4x^2 + 2x + 5$

$$\frac{dy}{dx} = 8x + 2$$
$$\frac{d^2y}{dx^2} = 8$$

b $y = 5x^3 - 3x^2 + 2x - 50$

$$\frac{dy}{dx} = 15x^2 - 6x + 2$$
$$\frac{d^2y}{dx^2} = 30x - 6$$

c $y = (5x^3 - 3x^2)^3$

$$\begin{aligned}\frac{dy}{dx} &= 3(5x^3 - 3x^2)^2(15x^2 - 6x) \\ \frac{d^2y}{dx^2} &= 3[2(5x^3 - 3x^2)(15x^2 - 6x)(15x^2 - 6x) + (30x - 6)(5x^3 - 3x^2)^2] \\ &= 6(5x^3 - 3x^2)(15x^2 - 6x)^2 + 3(30x - 6)(5x^3 - 3x^2)^2\end{aligned}$$

d $y = 5xe^{3x^2 - 2x}$

$$
\begin{aligned}
\frac{dy}{dx} &= 5e^{3x^2 - 2x} + e^{3x^2 - 2x}(6x-2) \times 5x \\
&= 5e^{3x^2 - 2x} + e^{3x^2 - 2x}(30x^2 - 10x) \\
\frac{d^2y}{dx^2} &= 5e^{3x^2 - 2x}(6x-2) + e^{3x^2 - 2x}(6x-2)(30x^2 - 10x) + (60x - 10)e^{3x^2 - 2x}
\end{aligned}
$$

e $y = 20(100x + 20x^2 - x^3) - 500x$

$$
\begin{aligned}
\frac{dy}{dx} &= 20(100 + 40x - 3x^2) - 500 \\
\frac{d^2y}{dx^2} &= 20(40 - 6x) = 800 - 120x
\end{aligned}
$$

Problem 3. In the following problems you are given a production function for a firm where y is the level of output and x is the level of the variable input. You are given the price (p)of the output and the price (w) of the single variable input. For each problem write down an equation that represents profit for the firm. Then maximize this function by taking its derivative with respect to the variable input x and set equal to zero. What is the optimal level of x, of output, of cost, of profit?

a

$$output\,price \;=\; p \;= 20$$
$$input\,price \;=\; w \;= 500$$
$$y \;=\; output \;=\; f(x) \;=\; 100x \;+\; 20x^2 \;-\; x^3$$

For this example,

$$\begin{aligned} Profit &= 20(100x \;+\; 20x^2 \;-\; x^3) \;-\; 500x \\ &= 2000x \;+\; 400x^2 \;-\; 20x^3 \;-\; 500x \\ &= 1500x \;+\; 400x^2 \;-\; 20x^3 \\ \frac{d\,Profit}{dx} &= 1500 \;+\; 800x \;-\; 60x^2 \end{aligned}$$

$$output\, price \;=\; p \;= 4$$

$$input\, price \;=\; w \;= 4504$$

$$y \;=\; output \;=\; f(x) \;=\; 100x \;+\; 100x^2 \;-2x^3$$

$$\begin{aligned}
profit &= p \times y - w \times x \\
&= 4(100x + 100x^2 - 2x^3) - 4504x \\
&= -8x^3 + 400x^2 - 4104x \\
\frac{d\, profit}{dx} &= -24x^2 + 800x - 4104 = 0 \\
&\Rightarrow -8(3x - 19)(x - 27) = 0 \\
&\Rightarrow x = \frac{19}{3} \text{ or } x = 27 \\
\frac{d^2\, profit}{dx^2} &= -48x + 800 \\
\text{At } x &= \frac{19}{3}, \frac{d^2\, profit}{dx^2}(x) = 496 > 0 \\
\text{So, } x &= \frac{19}{3} \text{ is a local minimum.} \\
\text{At } x &= 27, \frac{d^2\, profit}{dx^2}(x) = -496 < 0 \\
\text{So, } x &= 27 \text{ is a local maximum.} \\
& \text{So, optimal level of x is 27.} \\
100 \times 27 + 100 \times 27^2 - 2 \times 27^3 &= 36243 \text{ is the optimal level of output.} \\
4504 \times 27 &= 121608 \text{ is the optimal level of cost.} \\
4 \times 36243 - 121608 &= 23364 \text{ is the optimal level of profit .}
\end{aligned}$$

$$output\,price \;=\; p \;= 5$$

$$input\,price \;=\; w \;= 6495$$

$$y \;=\; output \;=\; f(x) \;=\; 500x \;+\; 100x^2 \;-3x^3$$

$$\begin{aligned}
profit &= p \times y - w \times x \\
&= 5(500x + 100x^2 - 3x^3) - 6495x \\
&= -15x^3 + 500x^2 - 3995x \\
\frac{d\,profit}{dx} &= -45x^2 + 1000x - 3995 = 0 \\
&\Rightarrow -5(9x - 47)(x - 17) = 0 \\
&\Rightarrow x = \frac{47}{9} \text{ or } x = 17 \\
\frac{d^2\,profit}{dx^2} &= -90x + 1000 \\
\text{At } x &= \frac{47}{9}, \frac{d^2\,profit}{dx^2}(x) = 530 > 0, \\
\text{so, } x &= \frac{47}{9} \text{ is a local minimum.} \\
\text{At } x &= 27, \frac{d^2\,profit}{dx^2}(x) = -530 < 0, \\
\text{so, } x &= 17 \text{ is a local maximum.} \\
& \quad \text{So, optimal level of x is 17.} \\
500 \times 17 + 100 \times 17^2 - 3 \times 17^3 &= 22661 \text{ is the optimal level of output .} \\
6495 \times 17 &= 110415 \text{ is the optimal level of cost.} \\
5 \times 22661 - 110415 &= 2890 \text{ is the optimal level of profit.}
\end{aligned}$$

Problem 4. For each of the following problems you are given a price (p) and the cost function for a competitive firm. TC stands for total cost and y represents the level of output. Marginal cost is the derivative of the cost function with respect to output. Find the profit maximizing level of output for each case.

a

$$price \;=\; p \;= 475$$

$$Total\ Cost \;=\; TC \;= 500 \;+\; 100y \;-\; 10y^2 \;+\; y^3$$

$$\begin{aligned}
\text{Profit} &= \text{total revenue - total cost} \\
&= py - TC \\
&= 475y - (500 + 100y - 10y^2 + y^3) \\
&= -y^3 + 10y^2 + 375y - 500 \\
\frac{d\,profit}{dy} &= -3y^2 + 20y + 375 = 0 \\
&\Rightarrow -(3y+25)(y-15) = 0 \\
&\Rightarrow y = -\frac{25}{3}(Dropped), \text{ or } y = 15. \\
\frac{d^2\,profit}{dy^2} &= -6y + 20
\end{aligned}$$

At y=15, $\frac{d^2\,profit}{dy^2}(15) = -70 < 0$. So, y=15 is a local maximum.

So, profit maximizing level of output is 15.

b

$$p = 1341$$

$$TC = 1000 + 600y - 30y^2 + 3y^3$$

$$\begin{aligned}
\text{Profit} &= \text{total revenue - total cost} \\
&= py - TC \\
&= 1341y - (1000 + 600y - 30y^2 + 3y^3) \\
&= -3y^3 + 30y^2 + 741y - 1000 \\
\frac{d\,profit}{dy} &= -9y^2 + 60y + 741 = 0 \\
&\Rightarrow -3(3y + 19)(y - 13) = 0 \\
&\Rightarrow y = -\frac{19}{3}(Dropped), \text{ or } y = 13. \\
\frac{d^2\,profit}{dy^2} &= -18y + 60
\end{aligned}$$

At y =13, $\frac{d^2\,profit}{dy^2}(13) = -174 < 0$. So, y=13 is a local maximum.

So, profit maximizing level of output is 13.

$$p = 600$$

$$TC = 600 + 200y - 20y^2 + y^3$$

$$\begin{aligned}
\text{Profit} &= \text{total revenue - total cost} \\
&= py - TC \\
&= 600y - (600 + 200y - 20y^2 + y^3) \\
&= -y^3 + 20y^2 + 400y - 600 \\
\frac{d\,profit}{dy} &= -3y^2 + 40y + 400 = 0 \\
&\Rightarrow -(3y + 20)(y - 20) = 0 \\
&\Rightarrow y = -\frac{20}{3}(Dropped),\ \text{or } y = 20. \\
\frac{d^2\,profit}{dy^2} &= -6y + 40
\end{aligned}$$

At y = 20, $\frac{d^2\,profit}{dy^2}(20) = -80 < 0$. So, y=20 is a local maximum.

So, profit maximizing level of output is 20.

Problem 5. Solve the following systems of equations.

$$324x_1^{-4/5}x_2^{2/5} - 16 = 0$$
$$648x_1^{1/5}x_2^{-3/5} - 243 = 0$$

Take the ratio of these two equations, we have,

$$\begin{aligned}
\frac{324x_1^{-4/5}x_2^{2/5}}{648x_1^{1/5}x_2^{-3/5}} &= \frac{16}{243} \Rightarrow \frac{x_2}{2x_1} = \frac{16}{243} \Rightarrow x_2 = \frac{32}{243}x_1 \\
Substitute\ x_2 &= \frac{32}{243}x_1 \text{ into the first equation, we obtain,} \\
324x_1^{-4/5}(\frac{32}{243}x_1)^{2/5} &= 16 \\
&\Rightarrow 324x_1^{-4/5}(\frac{2^5}{3^5}x_1)^{2/5} = 16 \\
&\Rightarrow 324(\frac{2^2}{3^2})x_1^{-2/5} = 16 \\
&\Rightarrow x_1^{-2/5} = \frac{16}{144} = \frac{1}{9} = 3^{-2} \\
&\Rightarrow x_1 = (3^{-2})^{-5/2} = 3^5 = 243 \\
x_2 &= \frac{32}{243}x_1 = \frac{32}{243} \times 243 = 32 \\
\text{So, } x_1 &= 243, x_2 = 32.
\end{aligned}$$

Problem 6. Find the indefinite integral of each of the following functions. Write in the form F(x) + c.

a $f(x) \ = \ 9x^2 \ + \ 4x \ + \ 3$

$$\begin{aligned} \int f(x)\,dx &= \int (9x^2 \ + \ 4x \ + \ 3)\,dx \\ &= 3x^3 + 2x^2 + 3x + c \end{aligned}$$

b $f(x) \ = \ -9450 \ + \ 2160x \ - \ 54x^2$

$$\begin{aligned} \int f(x)\,dx &= \int (-9450 \ + \ 2160x \ - \ 54x^2)\,dx \\ &= -9450x + 1080x^2 - 18x^3 + c \end{aligned}$$

c $y \ = \ 5x^{-1/2}$

$$\begin{aligned} \int f(x)\,dx &= \int (5x^{-1/2})\,dx \\ &= 10x^{1/2} + c \end{aligned}$$

d $y = -\frac{5}{2x^{3/2}}$

$$\begin{aligned} \int f(x)\,dx &= \int (-\frac{5}{2x^{3/2}})\,dx \\ &= \int (-\frac{5}{2}x^{-3/2})\,dx \\ &= 5x^{-1/2} + c \end{aligned}$$

e $y = 3x^{-1/2}z^{1/3} - 3$

$$\begin{aligned} \int f(x)\,dx &= \int (3x^{-1/2}z^{1/3} - 3)\,dx \\ &= 6x^{1/2}z^{1/3} - 3x + c \end{aligned}$$

Problem 7. Solve the following system of equations for x_1, x_2, and x_3.
$\{x_1 = 3,\ x_2 = 2,\ x_3 = 1\}$

$$x_1 - x_2 + 4x_3 = 5$$
$$-2x_1 + 3x_2 - 10x_3 = -10$$
$$2x_1 + 3x_2 - x_3 = 11$$

Two times the first equation, then plus the second equation, we have,

$$x_2 - 2x_3 = 0$$
$$\Rightarrow x_2 = 2x_3$$

Second equation plus the third equation, we obtain,

$$6x_2 - 11x_3 = 1$$

Substitute $x_2 = 2x_3$ into the above equation, we get,

$$6 \times 2x_3 - 11x_3 = 1 \Rightarrow x_3 = 1$$
$$x_2 = 2x_3 = 2 \times 1 = 2$$

Substitute $x_3 = 1, x_2 = 2$ into the first equation, we obtain,

$$x_1 - 2 + 4 \times 1 = 5 \Rightarrow x_1 = 5 + 2 - 4 = 3$$
$$So, x_1 = 3,\ x_2 = 2,\ x_3 = 1$$