ECONOMICS 207
SPRING 2006
LABORATORY EXERCISE 7

Problem 1. Find the second derivative of each of the following functions with respect to x

a $y = -600x + 200x^2 - 8x^3$

$$\begin{aligned}\frac{dy}{dx} &= -600 + 400x - 24x^2 \\ \frac{d^2y}{dx^2} &= 400 - 48x\end{aligned}$$

b $y = 480x^{3/5}z^{1/4} - 64x - 405z$

$$\begin{aligned}\frac{dy}{dx} &= 288x^{-2/5}z^{1/4} - 64 \\ \frac{d^2y}{dx^2} &= \frac{576}{5}x^{-7/5}z^{1/4}\end{aligned}$$

c $y = (4x^2 - 3x)^3$

$$\begin{aligned}\frac{dy}{dx} &= 3(4x^2 - 3x)^2(8x-3) \\ \frac{d^2y}{dx^2} &= 6(4x^2 - 3x)(8x-3)(8x-3) + 24(4x^2 - 3x)^2 \\ &= 6(4x^2 - 3x)(8x-3)^2 + 24(4x^2 - 3x)^2\end{aligned}$$

Date: 19 February 2007.

Problem 2. Find the definite integral of each of the following functions.

a $\int_1^5 (4x + 3)\, dx$

$$\begin{aligned}\int_1^5 (4x + 3)\, dx &= (2x^2 + 3x)|_1^5 = (2 \times 5^2 + 3 \times 5) - (2 \times 1^2 + 3 \times 1) \\ &= 65 - 5 = 60\end{aligned}$$

b $\int_{32}^{243} \left(288x^{-2/5}z^{1/4} - 64\right) dx, \quad z = 256.$

$$\begin{aligned}\int_{32}^{243} \left(288x^{-2/5}z^{1/4} - 64\right) dx &= \int_{32}^{243} \left(288x^{-2/5}256^{1/4} - 64\right) dx = \int_{32}^{243} \left(1152x^{-2/5} - 64\right) dx \\ &= (1920x^{3/5} - 64x)|_{32}^{243} \\ &= (51840 - 15552) - (15360 - 2048) = 36288 - 13312 = 22976\end{aligned}$$

c $\int_0^9 \left(6x^2 \;-\; 20x \;+\; 100\right)\,dx$

$$\begin{aligned}\int_0^9 \left(6x^2 \;-\; 20x \;+\; 100\right)\,dx &= (2x^3 - 10x^2 + 100x)|_0^9 \\ &= (2 \times 9^3 - 10 \times 9^2 + 100 \times 9) - 0 \\ &= 1458 - 810 + 900 = 1548\end{aligned}$$

d $\int_{100}^{300} \left(150 \;-\; \frac{1}{2}x\right)\,dx$

$$\begin{aligned}\int_{100}^{300} \left(150 \;-\; \frac{1}{2}x\right)\,dx &= (150x - \frac{1}{4}x^2)|_{100}^{300} \\ &= (150 \times 300 - \frac{1}{4} \times 300^2) - (150 \times 100 - \frac{1}{4} \times 100^2) \\ &= (45000 - 22500) - (15000 - 2500) \\ &= 22500 - 12500 = 10000\end{aligned}$$

Problem 3. Solve the following systems of equations.

$$288x_1^{-2/5}x_2^{1/4} - 64 = 0$$
$$120x_1^{3/5}x_2^{-3/4} - 405 = 0$$

Take the ratio of these two equations, we have :

$$\frac{288x_1^{-2/5}x_2^{1/4}}{120x_1^{3/5}x_2^{-3/4}} = \frac{64}{405}$$
$$\Rightarrow \frac{x_2}{x_1} = \frac{16}{243} \Rightarrow x_2 = \frac{16}{243}x_1 = \frac{2^4}{3^5}x_1$$

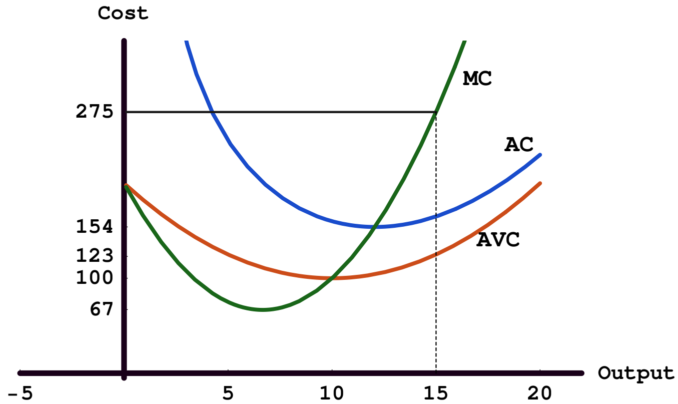
Substitute $x_2 = \frac{2^4}{3^5}x_1$ into the first equation, we have:

$$288x_1^{-2/5}(\frac{2^4}{3^5}x_1)^{1/4} = 64$$
$$\Rightarrow x_1^{-3/20} = \frac{64}{288} \times \frac{3^{5/4}}{2} = \frac{3^{5/4}}{9} = 3^{-3/4}$$
$$\Rightarrow x_1 = 3^{(-3/4)\times(-20/3)} = 3^5 = 243$$
$$x_2 = \frac{2^4}{3^5}x_1 = \frac{2^4}{3^5} \times 3^5 = 2^4 = 16$$

So, the answer is $x_1 = 243, x_2 = 16.$

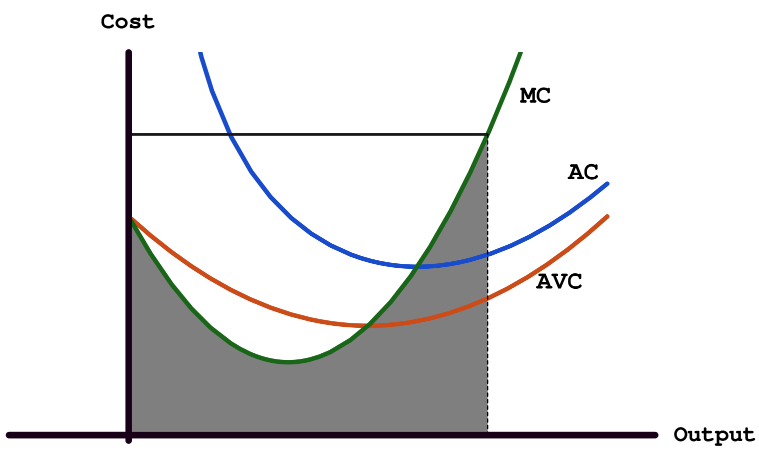
Problem 4. The cost function for a firm is a rule or mapping that tells the total cost of production of any output level produced by the firm. If the variable y represents the output of the firm, then the cost function is given by c(y). Marginal cost represents the change in the cost of production for the firm as output changes and is given by the derivative of the cost function with respect to output,i.e., Marginal Cost(MC) = $\frac{dc(y)}{dy}$. A competitive firm facing a fixed output price maximizes profit at the output level where marginal cost is equal to price as in the figure 1.

FIGURE 1. Profit Maximization



The area below the cost curve is a measure of variable cost and can be found by integrating the marginal cost curve from 0 to any given output level y. The shaded area in figure 2 represents the variable cost of production for the cost function $c(y) = 600 + 200y - 20\,y^2 + y^3$.

FIGURE 2. Variable Cost of Production and Producer Surplus



Producer surplus is the area below a given price and above the marginal cost curve. Producer surplus is the unshaded area below the horizontal line at 275 in figure 2. Producer surplus can be computed by subtracting the shaded area from total revenue.

a Find the profit maximizing level of output for the following firm. Demonstrate that the level you choose maximizes profit.

$$price \ = \ p \ = \$275$$

$$cost \ = \ c(y) \ = 600 \ + \ 200y \ - \ 20y^2 \ + \ y^3$$

$$\pi = P \times y - C(y) = 275y - 600 - 200y + 20y^2 - y^3$$

Take the first derivative of profit w.r.t y, and set it equal to 0.

$$\frac{d\pi}{dy} = 275 - 200 + 40y - 3y^2 = -3y^2 + 40y + 75 = 0$$

$$\Rightarrow (15 - y)(3y + 5) = 0 \Rightarrow y = 15 \text{ or } y = \frac{-5}{3}(Dropped)$$

Use the second order condition to check the property of this critical point.

$$\frac{d^2\pi}{dy^2} = -6y + 40$$

$$\text{At } y = 15, \pi^{''}(15) = -6 \times 15 + 40 = -50 < 0$$

$$\text{So, } y = 15 \text{ is a local maximum.}$$

b What is revenue minus variable cost for this firm when price is $275?

$$\begin{aligned} TR &= p \times y = 275 \times 15 = 4125 \\ VC &= 200 \times 15 - 20 \times 15^2 + 15^3 = 1875 \\ TR - VC &= 4125 - 1875 = 2250 \end{aligned}$$

c Find producer surplus for this firm assuming you are only given the following marginal cost function: $MC(y) = 200 - 40y + 3y^2$ and a price of $275.

$$\begin{aligned} VC &= \int_0^{15} MC(y)\, dy = \int_0^{15} 200 - 40y + 3y^2 \, dy \\ &= (200y - 20y^2 + y^3)|_0^{15} \\ &= (200 \times 15 - 20 \times 15^2 + 15^3) \\ &= 3000 - 4500 + 3375 \\ &= 1875 \\ PS &= TR - VC = 275 \times 15 - 1875 = 4125 - 1875 = 2250 \end{aligned}$$

Problem 5.

a Find the profit maximizing level of output for the following firm. Demonstrate that the level you choose maximizes profit.

$$price = p = \$500$$

$$cost = c(y) = 500 + 500y - 30y^2 + 2y^3$$

$$\begin{aligned}
\pi &= P \times y - C(y) = 500y - 500 - 500y + 30y^2 - 2y^3 \\
& \quad \text{Take the first derivative of profit w.r.t y, and set it equal to 0.} \\
\frac{d\pi}{dy} &= 60y - 6y^2 = 0 \\
&\Rightarrow 6y(10 - y) = 0 \Rightarrow y = 10 \text{ or } y = 0(Dropped) \\
& \quad \text{Use the second order condition to check the property of this critical point.} \\
\frac{d^2\pi}{dy^2} &= 60 - 12y \\
\text{At } y &= 10, \pi^{''}(10) = 60 - 120 = -60 < 0 \\
\text{So, } y &= 10 \text{ is a local maximum.}
\end{aligned}$$

b What is revenue minus variable cost for this firm when price is \$500?

$$\begin{aligned} TR &= p \times y = 500 \times 10 = 5000 \\ VC &= 500 \times 10 - 30 \times 10^2 + 2 \times 10^3 = 4000 \\ TR - VC &= 5000 - 4000 = 1000 \end{aligned}$$

c Find producer surplus for this firm assuming you are only given the following marginal cost function: $MC(y) = 500 - 60y + 6y^2$ and a price of \$500.

$$\begin{aligned} VC &= \int_0^{10} MC(y)\, dy = \int_0^{10} 500 - 60y + 6y^2\, dy \\ &= (500y - 30y^2 + 2y^3)|_0^{10} \\ &= (500 \times 10 - 30 \times 10^2 + 2 \times 10^3) \\ &= 5000 - 3000 + 2000 \\ &= 4000 \\ PS &= TR - VC = 500 \times 10 - 4000 = 1000 \end{aligned}$$

Problem 6.

a Find the profit maximizing level of output for the following firm. Demonstrate that the level you choose maximizes profit.

$$price \;=\; p \;= \$569$$

$$cost \;=\; c(y) \;= 300 \;+\; 200y \;-\; 20y^2 \;+\; 3y^3$$

$$\begin{aligned}
\pi &= P \times y - C(y) = 569y - 300 - 200y + 20y^2 - 3y^3 \\
& \quad \text{Take the first derivative of profit w.r.t y, and set it equal to 0.} \\
\frac{d\pi}{dy} &= 569 - 200 + 40y - 9y^2 = -9y^2 + 40y + 369 = 0 \\
&\Rightarrow (9-y)(9y+41) = 0 \Rightarrow y = 9 \text{ or } y = -\frac{41}{9}(Dropped) \\
& \quad \text{Use the second order condition to check the property of this critical point.} \\
\frac{d^2\pi}{dy^2} &= -18y + 40 \\
\text{At } y &= 9, \pi^{''}(9) = -162 + 40 = -122 < 0 \\
\text{So, } y &= 9 \text{ is a local maximum.}
\end{aligned}$$

b What is revenue minus variable cost for this firm when price is $569?

$$\begin{aligned} TR &= p \times y = 569 \times 9 = 5121 \\ VC &= 200 \times 9 - 20 \times 9^2 + 3 \times 9^3 = 2367 \\ TR - VC &= 5121 - 2367 = 2754 \end{aligned}$$

c Find producer surplus for this firm assuming you are only given the following marginal cost function: $MC(y) = 200 - 40y + 9y^2$ and a price of $569.

$$\begin{aligned} VC &= \int_0^9 MC(y)\,dy = \int_0^9 200 - 40y + 9y^2\,dy \\ &= (200y - 20y^2 + 3y^3)|_0^9 \\ &= (200 \times 9 - 20 \times 9^2 + 3 \times 9^3) \\ &= 1800 - 1620 + 2187 \\ &= 2367 \\ PS &= TR - VC = 569 \times 9 - 2367 = 2754 \end{aligned}$$

Problem 7. Solve the following system of equations for x_1, x_2, and x_3.
$\{x_1 = 3, x_2 = 2, x_3 = 1\}$

$$x_1 - x_2 + 4x_3 = 5$$
$$-2x_1 + 3x_2 - 10x_3 = -10$$
$$2x_1 + 3x_2 - x_3 = 11$$

Two times the first equation, then plus the second equation, we have,

$$x_2 - 2x_3 = 0$$
$$\Rightarrow x_2 = 2x_3$$

Second equation plus the third equation, we obtain,

$$6x_2 - 11x_3 = 1$$

Substitute $x_2 = 2x_3$ into the above equation, we get,

$$6 \times 2x_3 - 11x_3 = 1 \Rightarrow x_3 = 1$$
$$x_2 = 2x_3 = 2 \times 1 = 2$$

Substitute $x_3 = 1, x_2 = 2$ into the first equation, we obtain,

$$x_1 - 2 + 4 \times 1 = 5 \Rightarrow x_1 = 5 + 2 - 4 = 3$$
$$So, x_1 = 3, x_2 = 2, x_3 = 1$$

Problem 8. For each of the following problems, find the critical points. For each critical point state whether the function is at a relative maximum, relative minimum, or otherwise.

a $y = x^4$

$$\begin{aligned}
\frac{dy}{dx} &= 4x^3 = 0 \\
&\Rightarrow x = 0 \\
\frac{d^2y}{dx^2} &= 12x^2, \frac{d^3y}{dx^3} = 24x, \frac{d^4y}{dx^4} = 24 \\
At\ x &= 0,\ \frac{d^2y}{dx^2} = 12x^2 = 0, \frac{d^3y}{dx^3} = 24x = 0, and, \frac{d^4y}{dx^4} = 24 > 0 \\
So, at\ x &= 0\ the\ function\ is\ at\ a\ relative\ minimum.
\end{aligned}$$

b $y = 20x - 2x^2$

$$\begin{aligned}
\frac{dy}{dx} &= 20 - 4x = 0 \\
&\Rightarrow x = 5 \text{ is a critical point.} \\
\frac{d^2y}{dx^2} &= -4 \\
So, f''(5) &= -4 < 0 \\
So, at\ x &= 5\ the\ function\ is\ at\ a\ relative\ maximum.
\end{aligned}$$

c $f(x) = 576x + 30x^2 - 3x^3$

$$\begin{aligned}
f^{'}(x) &= 576 + 60x - 9x^2 = 0 \\
&\Rightarrow 3(12 - x)(3x + 16) \\
&\Rightarrow x = 12, x = \frac{-16}{3}, \text{ are two critical points.} \\
f^{''}(x) &= 60 - 18x \\
At\ x &= 12, f^{''}(12) = 60 - 18 \times 12 = -156 < 0 \\
At\ x &= \frac{-16}{3}, f^{''}(\frac{-16}{3}) = 60 - 18 \times \frac{-16}{3} = 156 > 0 \\
So, x &= 12\ is\ a\ relative\ maximum\ point. x = \frac{-16}{3}\ is\ a\ relative\ minimum\ point
\end{aligned}$$

d $f(x) = -36x - 25x^2 + 2x^3$

$$\begin{aligned}
f^{'}(x) &= -36 - 50x + 6x^2 = 0 \\
&\Rightarrow 6(x - 9)(x + \frac{2}{3}) \\
&\Rightarrow x = 9, x = \frac{-2}{3}, \text{ are two critical points.} \\
f^{''}(x) &= -50 + 12x \\
At\ x &= 9, f^{''}(9) = -50 + 9 \times 12 = 58 > 0 \\
At\ x &= \frac{-2}{3}, f^{''}(\frac{-2}{3}) = -50 + 12 \times \frac{-2}{3} = -58 < 0 \\
So, x &= \frac{-2}{3}\ is\ a\ relative\ maximum\ point. x = 9\ is\ a\ relative\ minimum\ point
\end{aligned}$$

e $f(x) = 200x - 50x^2 + 4x^3$

$$\begin{aligned}
f^{'}(x) &= 200 - 100x + 12x^2 = 0 \\
&\Rightarrow 12(x-5)(x-\frac{10}{3}) \\
&\Rightarrow x = 5, x = \frac{10}{3},\ \text{are two critical points.} \\
f^{''}(x) &= -100 + 24x \\
At\ x &= 5, f^{''}(5) = -100 + 24 \times 5 = 20 > 0 \\
At\ x &= \frac{10}{3}, f^{''}(\frac{10}{3}) = -100 + 24 \times \frac{10}{3} = -20 < 0 \\
So, x &= \frac{10}{3}\ is\ a\ relative\ maximum\ point. x = 5\ is\ a\ relative\ minimum\ point
\end{aligned}$$

f $f(x) = x^3 + 3x$

$$\begin{aligned}
f^{'}(x) &= 3x^2 + 3 = 0 \\
&\Rightarrow x^2 = -1 \quad \text{There is no root for this equation} \\
&\quad \text{So there does not exist a critical point for this function.}
\end{aligned}$$

g $f(x) = 3x^{5/3} - 5x$

$$\begin{aligned} f^{'}(x) &= 5x^{2/3} - 5 = 0 \\ &\Rightarrow x^{2/3} = 1 \\ &\Rightarrow x = 1,\ x = -1, are\ two\ critical\ points. \\ f^{''}(x) &= \frac{10}{3}x^{-1/3} \\ At\ x &= 1, f^{''}(1) = \frac{10}{3}(1)^{-1/3} = \frac{10}{3} > 0 \\ At\ x &= -1, f^{''}(-1) = \frac{10}{3}(-1)^{-1/3} = -\frac{10}{3} < 0 \\ So, x &= -1\ is\ a\ relative\ maximum\ point. x = 1\ is\ a\ relative\ minimum\ point \end{aligned}$$

h $f(x) = 42x^2 - 4x^3 - 72x$

$$\begin{aligned} f^{'}(x) &= 84x - 12x^2 - 72 = 0 \\ &\Rightarrow -12(x-1)(x-6) \\ &\Rightarrow x = 1, x = 6,\ \text{are two critical points.} \\ f^{''}(x) &= 84 - 24x \\ At\ x &= 1, f^{''}(1) = 84 - 24 = 60 > 0 \\ At\ x &= 6, f^{''}(6) = 84 - 24 \times 6 = -60 < 0 \\ So, x &= 6\ is\ a\ relative\ maximum\ point. x = 1\ is\ a\ relative\ minimum\ point \end{aligned}$$

i $f(x) = x^5$

$$\begin{aligned} f^{'}(x) &= 5x^4 = 0 \\ &\Rightarrow x = 0 \\ f^{''}(x) &= 20x^3, f^{'''}(x) = 60x^2, f^{''''}(x) = 120x, f^{'''''}(x) = 120 \\ At\, x &= 0, f^{''}(0) = 0, f^{'''}(0) = 0, f^{''''}(0) = 0, f^{'''''}(0) = 120 > 0 \\ So, x &= 0\, is\, an\, inflection\, point. \end{aligned}$$

j $f(x) = 100x^{1/2} - 2x$

$$\begin{aligned} f^{'}(x) &= 50x^{-1/2} - 2 = 0 \\ &\Rightarrow 50x^{-1/2} = 2 \Rightarrow x^{-1/2} = \frac{1}{25} \Rightarrow x = 625 \\ f^{''}(x) &= -25x^{-3/2} \\ At\, x &= 625, f^{''}(625) = -25 \times (625)^{-3/2} < 0 \\ So, x &= 625\, is\, a\, relative\, maximum\, point. \end{aligned}$$

Problem 9. In the following problem you are given a production function for a firm where y is the level of output and x is the level of the variable input. You are given the price (p) of the output and the price (w) of the single variable input. Write down an equation that represents profit for the firm. Then maximize this function by taking its derivative with respect to the variable input x and set equal to zero. What is the optimal level of x? Show why this x is the one one that maximizes profit.

$$output\ price \ = \ p \ = 4$$

$$input\ price \ = \ w \ = 1000$$

$$y \ = \ output \ = \ f(x) \ = \ 100x \ + \ 50x^2 \ - 2x^3$$

$$\begin{aligned}
profit &= total\ revenue - total\ cost \\
\pi &= p \times y - w \times x, \ \ y = 100x + 50x^2 - 2x^3 \\
So, \pi &= 4 \times (100x + 50x^2 - 2x^3) - 1000 \times x = -8x^3 + 200x^2 - 600x \\
\frac{d\pi}{dx} &= -24x^2 + 400x - 600 = 0 \\
&\Rightarrow -8(x-15)(3x-5) = 0 \\
&\Rightarrow x = 15, x = \frac{5}{3}\ are\ two\ critical\ points. \\
\frac{d^2\pi}{dx^2} &= -48x + 400 \\
At\ x &= 15,\ \pi^{''}(15) = -48 \times 15 + 400 = -320 < 0 \\
At\ x &= \frac{5}{3},\ \pi^{''}(\frac{5}{3}) = -48 \times \frac{5}{3} + 400 = 320 > 0 \\
&\Rightarrow At\ x = 15, the\ profit\ function\ is\ at\ a\ relative\ maximum. \\
At\ x &= \frac{5}{3}, the\ profit\ function\ is\ at\ a\ relative\ minimum.
\end{aligned}$$