ECONOMICS 207
SPRING 2006
LABORATORY EXERCISE 8 KEY

Problem 1. Consider the following matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 4 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

$$F = \begin{bmatrix} -2 & -2 & 4 \\ -3 & -1 & 4 \\ -6 & -6 & 10 \end{bmatrix} \quad G = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 9 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \quad c = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Compute the following

a A + C

$$A + C = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$$

b A + 2C

$$A + 2C = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} + 2 \times \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 6 & 5 \end{bmatrix}$$

c A + B

$A + B$ is not defined. Matrix dimensions do not agree.

Date: 28 December 2006.

d AC

$$A \times C = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 8 & 3 \end{bmatrix}$$

e a′

$$a\prime = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}' = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

f a′b

$$a\prime b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}' \times \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \end{bmatrix}$$

g CA

$$CA = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}$$

h Ba

$$B \times a = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 4 & 5 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 22 \\ 26 \end{bmatrix}$$

i Bb

$$B \times b = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 4 & 5 \end{bmatrix} \times \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 17 \end{bmatrix}$$

j c'a

$$c\prime = \begin{bmatrix} 1 \\ 4 \end{bmatrix}' = \begin{bmatrix} 1 & 4 \end{bmatrix}, \text{which is } 1 \times 2$$

$$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \text{Which is } 3 \times 1.$$

Matrix dimensions do not agree. So, $c\prime \times a$ is not defined

k c'B

$$c\prime B = \begin{bmatrix} 1 \\ 4 \end{bmatrix}' \times \begin{bmatrix} 2 & 1 & 6 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 4 \end{bmatrix} \times \begin{bmatrix} 2 & 1 & 6 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 14 & 17 & 26 \end{bmatrix}$$

1 FG

$$Let\ FG = \begin{bmatrix} -2 & -2 & 4 \\ -3 & -1 & 4 \\ -6 & -6 & 10 \end{bmatrix} \times \begin{bmatrix} 4 & 2 & 0 \\ 2 & 9 & 0 \\ 0 & 0 & 2 \end{bmatrix} = V$$

We obtain the first element of the product by multiplying the first row of F by the first column of G.

$$V_{11} = \begin{bmatrix} -2 & -2 & 4 \end{bmatrix} \times \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = -12,$$

We obtain the second element of the first row of the product by multiplying the first row of F by the second column of G.

$$V_{12} = \begin{bmatrix} -2 & -2 & 4 \end{bmatrix} \times \begin{bmatrix} 2 \\ 9 \\ 0 \end{bmatrix} = -22$$

We obtain the 3^{rd} element of the first row of the product by multiplying the first row of F by the 3^{rd} column of G.

$$V_{13} = \begin{bmatrix} -2 & -2 & 4 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = 8$$

Similarly, we can obtain the second and the third row of the product, which are $\begin{bmatrix} -14 & -15 & 8 \end{bmatrix}, and \begin{bmatrix} -36 & -66 & 20 \end{bmatrix}$

$$So, FG = \begin{bmatrix} -12 & -22 & 8 \\ -14 & -15 & 8 \\ -36 & -66 & 20 \end{bmatrix}$$

m Fa

$$Fa = \begin{bmatrix} -2 & -2 & 4 \\ -3 & -1 & 4 \\ -6 & -6 & 10 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$

n b′G

$$b'G = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}' \times \begin{bmatrix} 4 & 2 & 0 \\ 2 & 9 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 1 \end{bmatrix} \times \begin{bmatrix} 4 & 2 & 0 \\ 2 & 9 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 27 & 2 \end{bmatrix}$$

o A + 2D

$$A + 2D = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} + 2 \times \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 8 & 4 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 6 & 7 \end{bmatrix}$$

p B′D

$$B'D = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 4 & 5 \end{bmatrix}' \times \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 6 & 5 \end{bmatrix} \times \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 10 \\ 12 & 10 \\ 34 & 22 \end{bmatrix}$$

q BG

$$BG = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 4 & 5 \end{bmatrix} \times \begin{bmatrix} 4 & 2 & 0 \\ 2 & 9 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 13 & 12 \\ 20 & 42 & 10 \end{bmatrix}$$

Problem 2. For each of the following problems, find the critical points. For each critical point state whether the function is at a relative maximum, relative minimum, or otherwise. Also find the points of inflection for each function.

a $y = 20x^{1/2} - 2x$

Set the first order condition equal to 0 to solve the critical points:

$$\begin{aligned} \frac{\partial y}{\partial x} &= 10x^{-1/2} - 2 = 0 \\ \Rightarrow & \ x^{-1/2} = \frac{1}{5} \Rightarrow x = 25 \end{aligned}$$

Use the second order condition to check the property of the critical points:

$$\begin{aligned} \frac{\partial^2 y}{\partial x^2} &= -5x^{-3/2} \\ \text{For } x &= 25, \quad f''(25) = -5(25)^{-3/2} < 0, \text{ so, } x = 25 \text{ is a relative maximum} \end{aligned}$$

b $y = x^4 - 4x^3 + 10$

Set the first order condition equal to 0 to solve the critical points:

$$\begin{aligned} \frac{\partial y}{\partial x} &= 4x^3 - 12x^2 = 0 \\ \Rightarrow & \ 4x^2(x-3) = 0 \\ \Rightarrow & \ x = 0 \quad or \quad x = 3 \end{aligned}$$

Use the second order condition to check the property of the critical points:

$$\begin{aligned} \frac{\partial^2 y}{\partial x^2} &= 12x^2 - 24x \qquad \frac{\partial^3 y}{\partial x^3} = 24x - 24 \\ \text{For } x &= 0, \quad f''(0) = 0,\ f'''(0) = -24 < 0 \text{ so, x=0 is a inflection point.} \\ \text{For } x &= 3, \quad f''(3) = 36 > 0 \text{ so, x=3 is a relative minimum} \end{aligned}$$

c $f(x) = x^{5/3} - 5x^{2/3}$

Set the first order condition equal to 0 to solve the critical points:

$$\begin{aligned}\frac{\partial f(x)}{\partial x} &= \frac{5}{3}x^{\frac{2}{3}} - \frac{10}{3}x^{-\frac{1}{3}} = 0 \\ &\Rightarrow \frac{5}{3}x^{-\frac{1}{3}}(x-2) = 0 \\ &\Rightarrow x = 2\end{aligned}$$

Use the second order condition to check the property of the critical points:

$$\begin{aligned}\frac{\partial^2 f(x)}{\partial x^2} &= \frac{10}{9}x^{-\frac{1}{3}} - \frac{10}{9}x^{-\frac{4}{3}} \\ \text{For } x &= 2, \quad f''(2) = \frac{10}{9}(2)^{-\frac{1}{3}}(1-2^{-1}) = \frac{5}{9}(2)^{-\frac{1}{3}} > 0 \text{ so, x=2 is a relative minimum.} \\ \text{For } x &= 3, \quad f''(3) = 36 > 0 \text{ so, x=3 is a relative minimum}\end{aligned}$$

d $f(x) = \frac{15}{4}x^4 + \frac{2}{3}x^3 - 12x^2$

Set the first order condition equal to 0 to solve the critical points:

$$\begin{aligned}\frac{\partial f(x)}{\partial x} &= 15x^3 + 2x^2 - 24x = 0 \\ &\Rightarrow x(3x+4)(5x-6) = 0 \\ &\Rightarrow x = 0 \quad or \quad x = -\frac{4}{3} \quad or \quad x = \frac{6}{5}\end{aligned}$$

Use the second order condition to check the property of the critical points:

$$\begin{aligned}\frac{\partial^2 f(x)}{\partial x^2} &= 45x^2 + 4x - 24 \qquad \frac{\partial^3 f(x)}{\partial x^3} = 90x + 4 \\ \text{For } x &= 0, \quad f''(0) = -24 < 0, \ f'''(0) = 4.\text{so, x=0 is a relative maximum.} \\ \text{For } x &= -\frac{4}{3}, \quad f''(-\frac{4}{3}) = \frac{152}{3} > 0 \text{ so, x=} -\frac{4}{3} \text{ is a relative minimum} \\ \text{For } x &= \frac{6}{5}, \quad f''(\frac{6}{5}) = \frac{228}{5} > 0 \text{ so, x=}\frac{6}{5} \text{ is a relative minimum}\end{aligned}$$

Problem 3. Find the definite integral of each of the following functions.

a $\int_2^5 \left(4x^2 + 3x - 2\right) dx$

$$\begin{aligned}\int_2^5 \left(4x^2 + 3x - 2\right) dx &= \left(\frac{4}{3}x^3 + \frac{3}{2}x^2 - 2x\right)|_2^5 \\ &= \left(\frac{4}{3}(5)^3 + \frac{3}{2}(5)^2 - 2(5)\right) - \left(\frac{4}{3}(2)^3 + \frac{3}{2}(2)^2 - 2(2)\right) \\ &= \frac{1165}{6} - \frac{38}{3} = \frac{1165-76}{6} = \frac{1089}{6} = \frac{363}{2}\end{aligned}$$

b $\int_{16}^{81} \left(3x^{-3/4}z^{2/3} - 1\right) dx, \quad z = 27.$

$$\begin{aligned}\int_{16}^{81} \left(3x^{-3/4}z^{2/3} - 1\right) dx &= \int_{16}^{81} \left(3x^{-3/4}(27)^{2/3} - 1\right) dx = \int_{16}^{81} \left(27x^{-3/4} - 1\right) dx \\ &= (4 \times 27x^{1/4} - x)|_{16}^{81} = (4 \times 27 \times 3 - 81) - (4 \times 27 \times 2 - 16) \\ &= 243 - 200 = 43\end{aligned}$$

c $\int_0^{11} \left(6x^2 \; - \; 48x \; + \; 250\right) \, dx$

$$\begin{aligned} \int_0^{11} \left(6x^2 \; - \; 48x \; + \; 250\right) \, dx &= \left(2x^3 \; - \; 24x^2 \; + \; 250x\right)|_0^{11} \\ &= \left(2(11)^3 \; - \; 24(11)^2 \; + \; 250(11)\right) - 0 \\ &= 2508 \end{aligned}$$

d $\int_2^4 e^{3x} \, dx$

$$\begin{aligned} \int_2^4 e^{3x} \, dx &= \int_2^4 \frac{1}{3} \, e^{3x} \, d3x = \frac{1}{3} \int_2^4 e^{3x} \, d3x = \frac{1}{3} e^{3x}|_2^4 \\ &= \frac{1}{3}[(e^{3\times4}) - (e^{3\times2})] = \frac{1}{3}[e^{12} - e^6] \end{aligned}$$

Problem 4. Solve the following systems of equations.

$$3x_1^{-3/4}x_2^{2/3} - 1 = 0$$

$$8x_1^{1/4}x_2^{-1/3} - 8 = 0$$

Take the ratio of these two equations :

$$\frac{3x_1^{-3/4}x_2^{2/3}}{8x_1^{1/4}x_2^{-1/3}} = \frac{1}{8}$$

$$\Rightarrow \frac{x_2}{x_1} = \frac{1}{3} \Rightarrow x_1 = 3x_2$$

Substituting $x_1 = 3x_2$ into the second equation, we have:

$$8(3x_2)^{1/4}x_2^{-1/3} - 8 = 0$$

$$\Rightarrow 3^{\frac{1}{4}}x_2^{-\frac{1}{12}} = 1 \Rightarrow x_2^{\frac{-1}{12}} = 3^{-\frac{1}{4}}$$

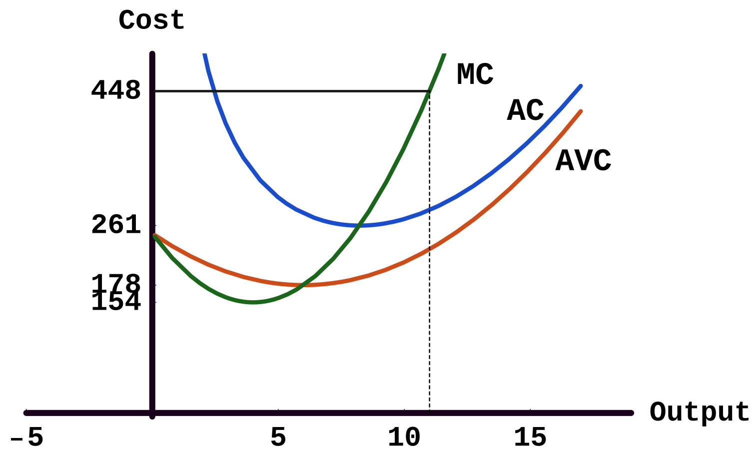
$$\Rightarrow x_2 = (3^{-\frac{1}{4}})^{-12} = 3^3 = 27$$

$$x_1 = 3x_2 = 3 \times 27 = 81$$

$$So,\ x_1 = 81, \qquad x_2 = 27$$

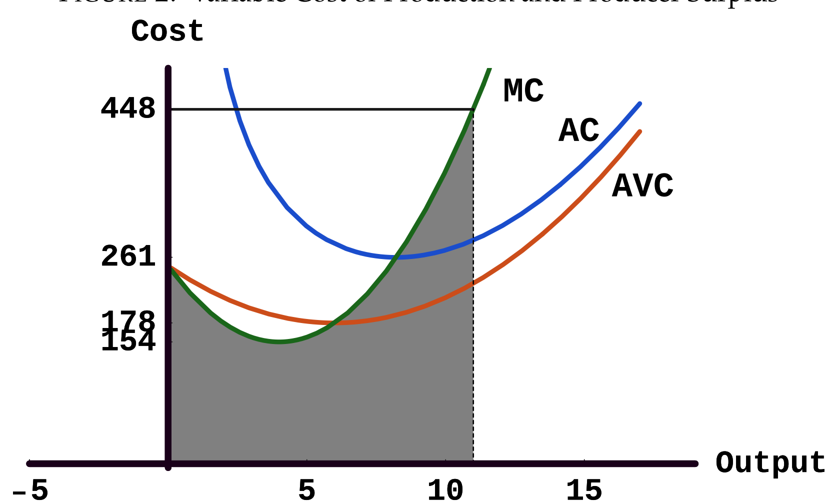
Problem 5. The cost function for a firm is a rule or mapping that tells the total cost of production of any output level produced by the firm. If the variable y represents the output of the firm, then the cost function is given by c(y). Marginal cost represents the change in the cost of production for the firm as output changes and is given by the derivative of the cost function with respect to output,i.e., Marginal Cost(MC) = $\frac{dc(y)}{dy}$. A competitive firm facing a fixed output price maximizes profit at the output level where marginal cost is equal to price as in the figure 1.

FIGURE 1. Profit Maximization



The area below the cost curve is a measure of variable cost and can be found by integrating the marginal cost curve from 0 to any given output level y. The shaded area in figure 2 represents the variable cost of production for the cost function $c(y) = 600 + 250y - 24\,y^2 + 2y^3$.

FIGURE 2. Variable Cost of Production and Producer Surplus



Producer surplus is the area below a given price and above the marginal cost curve. Producer surplus is the unshaded area below the horizontal line at 448 in figure 2. Producer surplus can be computed by subtracting the shaded area from total revenue.

a Find the profit maximizing level of output for the following firm. Demonstrate that the level you choose maximizes profit.

$$price \;=\; p \;= \$448$$

$$cost \;=\; c(y) \;= 600 \;+\; 250y \;-\; 24y^2 \;+\; 2y^3$$

$$\begin{aligned}
\pi(y) &= py - c(y) = 448y - (600 + 250y - 24y^2 + 2y^3) = -2y^3 + 24y^2 + 198y - 600 \\
&\quad Take\ the\ derivative\ of\ the\ profit\ function\ with\ respect\ to\ y\ and\ set\ equal\ to\ zero, \\
\frac{\partial \pi}{\partial y} &= -6y^2 + 48y + 198 = 0 \Rightarrow -6(x+3)(x-11) = 0 \\
&\Rightarrow y = -3\ (Dropped)\ and\ y = 11\ are\ the\ critical\ points. \\
&\quad Use\ the\ second\ order\ condition\ to\ check\ the\ property\ of\ the\ critical\ points, \\
\frac{\partial^2 \pi}{\partial y^2} &= -12y + 48 \\
\pi''(11) &= -12 \times 11 + 48 = -84 < 0 \\
&\Rightarrow y = 11\ maximizes\ profit.
\end{aligned}$$

b What is revenue minus variable cost for this firm when price is \$448? This is the same as producer surplus.

$$\begin{aligned} revenue &= py = 448 \times 11 = 4928 \\ variable\ cost &= 250 \times 11 - 24(11)^2 + 2(11)^3 = 2508 \\ producer\ surplus &= revenue - variable\ cost \\ &= 4928 - 2508 = 2420 \end{aligned}$$

c Find producer surplus for this firm assuming you are only given the following marginal cost function: $MC(y) = 250 - 48y + 6y^2$ and a price of \$448.

$$\begin{aligned} revenue &= py = 448 \times 11 = 4928 \\ variable\ cost &= \int_0^{11} 250 - 48y + 6y^2 dy = (250y - 24y^2 + 2y^3)|_0^{11} \\ &= \big(2(11)^3 - 24(11)^2 + 250(11)\big) - 0 \\ &= 2508 \\ producer\ surplus &= revenue - variable\ cost \\ &= 4928 - 2508 = 2420 \end{aligned}$$

Problem 6. In the following problem you are given a production function for a firm where y is the level of output and x is the level of the variable input. You are given the price (p) of the output and the price (w) of the single variable input. Write down an equation that represents profit for the firm. Then maximize this function by taking its derivative with respect to the variable input x and set equal to zero. What is the optimal level of x? Show why this x is the one one that maximizes profit.

$$output\ price \ = \ p \ = 2$$

$$input\ price \ = \ w \ = 300$$

$$y \ = \ output \ = \ f(x) \ = \ 50x^2 \ - 2x^3$$

$$\pi(x) = p \times y - w \times x = 2(50x^2 - 2x^3) - 300x = -4x^3 + 100x^2 - 300x$$

$$\begin{aligned}
& Take\ the\ derivative\ of\ the\ profit\ function\ with\ respect\ to\ x\ and\ set\ equal\ to\ zero, \\
\frac{\partial \pi}{\partial x} &= -12x^2 + 200x - 300 = 0 \Rightarrow -2(2x-30)(3x-5) = 0 \\
&\Rightarrow x = 15\ and\ x = \frac{5}{3}\ are\ the\ critical\ points. \\
& Use\ the\ second\ order\ condition\ to\ check\ the\ property\ of\ the\ critical\ points, \\
\frac{\partial^2 \pi}{\partial x^2} &= -24x + 200 \\
At\ x &= 15, \pi''(15) = -24 \times 15 + 200 = -160 < 0. \text{ So, x=15 is a local maximum.} \\
At\ x &= \frac{5}{3}, \pi''(\frac{5}{3}) = -24 \times \frac{5}{3} + 200 = 160 > 0. \text{So, x=}\frac{5}{3} \text{ is a local minimum.} \\
So, x &= 15\ \ maximizes\ profit.
\end{aligned}$$

Problem 7. Solve the following system of equations for x_1, x_2, and x_3.
$\{x_1 = 2,\ x_2 = -2,\ x_3 = 3\}$

$$x_1 - x_2 - x_3 = 1$$
$$-3x_1 + 4x_2 + 2x_3 = -8$$
$$2x_1 + x_2 - 4x_3 = -10$$

$$\begin{aligned}
Let \quad A &= \begin{bmatrix} 1 & -1 & -1 \\ -3 & 4 & 2 \\ 2 & 1 & -4 \end{bmatrix}, \qquad a = \begin{bmatrix} 1 \\ -8 \\ -10 \end{bmatrix}, \\
then\ x_1, x_2, and\ x_3\ is\ the\ solustion\ of\ Ax &= a. \\
&\Rightarrow x = A^{-1} \times a \\
A^{-1} &= \frac{1}{Det(A)} Adj(A) \\
Det(A) &= 1 \times (-1)^2 \begin{vmatrix} 4 & 2 \\ 1 & -4 \end{vmatrix} + (-1) \times (-1)^3 \begin{vmatrix} -3 & 2 \\ 2 & -4 \end{vmatrix} \\
&\quad +(-1) \times (-1)^4 \begin{vmatrix} -3 & 4 \\ 2 & 1 \end{vmatrix} \\
&= (-16-2) + (12-4) - (-3-8) = -18 + 8 + 11 = 1 \\
Adj(A) &= \begin{bmatrix} \begin{vmatrix} 4 & 2 \\ 1 & -4 \end{vmatrix} & -\begin{vmatrix} -1 & -1 \\ 1 & -4 \end{vmatrix} & \begin{vmatrix} -1 & -1 \\ 4 & 2 \end{vmatrix} \\ -\begin{vmatrix} -3 & 2 \\ 2 & -4 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 2 & -4 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ -3 & 2 \end{vmatrix} \\ \begin{vmatrix} -3 & 4 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ -3 & 4 \end{vmatrix} \end{bmatrix} \\
&= \begin{bmatrix} -18 & -5 & 2 \\ -8 & -2 & 1 \\ -11 & -3 & 1 \end{bmatrix} \\
A^{-1} &= \frac{1}{Det(A)} Adj(A) = \frac{1}{1} \begin{bmatrix} -18 & -5 & 2 \\ -8 & -2 & 1 \\ -11 & -3 & 1 \end{bmatrix} = \begin{bmatrix} -18 & -5 & 2 \\ -8 & -2 & 1 \\ -11 & -3 & 1 \end{bmatrix} \\
x &= A^{-1} \times a = \begin{bmatrix} -18 & -5 & 2 \\ -8 & -2 & 1 \\ -11 & -3 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ -8 \\ -10 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} \\
So,\ x_1 &= 2,\ x_2 = -2,\ x_3 = 3.
\end{aligned}$$