ECONOMICS 207
SPRING 2006
LABORATORY EXERCISE 9

Problem 1. For each of the following problems, find the critical points. For each critical point state whether the function is at a relative maximum, relative minimum, or otherwise. Check to see if there are points of inflection **at points other than** critical points. Verify that the prospective points of inflection are actually critical points.

a $f(x) = x^3 - 9x^2 - 48x + 52$

Set the first order condition equal to 0 to find the critical points:

$$\begin{aligned} \frac{\partial f(x)}{\partial x} &= 3x^2 - 18x - 48 = 0 \\ &\Rightarrow 3(x-8)(x+2) = 0 \\ &\Rightarrow x = 8 \quad or \quad x = -2 \end{aligned}$$

Use the second order condition to check the property of the critical points:

$$\begin{aligned} \frac{\partial^2 f(x)}{\partial x^2} &= 6x - 18 \\ For \quad x &= 8, \quad f''(x) = 6 \times 8 - 18 = 30 > 0 \Rightarrow x = 8 \quad \text{is} \quad \text{a local minimum.} \\ For \quad x &= -2, \quad f''(x) = 6 \times (-2) - 18 = -30 < 0 \Rightarrow x = -2 \quad \text{is} \quad \text{a local maximum.} \end{aligned}$$

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b $y = \frac{1}{2}x^6 + \frac{13}{5}x^5 - \frac{5}{2}x^4$

Set the first order condition equal to 0 to find the critical points:

$$\begin{aligned}\frac{\partial f(x)}{\partial x} &= 3x^5 + 13x^4 - 10x^3 = 0 \\ &\Rightarrow x^3(3x-2)(x+5) = 0 \\ &\Rightarrow x = 0 \quad or \quad x = \frac{2}{3} \quad or \quad x = -5\end{aligned}$$

Use the second order condition to check the property of the critical points:

$$\frac{\partial^2 f(x)}{\partial x^2} = 15x^4 + 52x^3 - 30x^2 \quad \frac{\partial^3 f(x)}{\partial x^3} = 60x^3 + 156x^2 - 60x \quad \frac{\partial^4 f(x)}{\partial x^4} = 180x^2 + 312x - 60$$

$For \quad x = 0, \quad f''(x) = 0, \quad f'''(x) = 0, \quad f''''(x) = -60 < 0$. So, $x = 0$ is a inflection point

$For \quad x = \frac{2}{3}, \quad f''(x) = 5.04 > 0 \Rightarrow x = \frac{2}{3}$ is a local minimum.

$For \quad x = -5, \quad f''(x) = 2125 > 0 \Rightarrow x = -5$ is a local minimum.

c $f(x) = \frac{4x}{x^2 + 4}$

Set the first order condition equal to 0 to find the critical points:

$$\begin{aligned}\frac{\partial f(x)}{\partial x} &= \frac{4(x^2 + 4) - 2x \times 4x}{(x^2 + 4)^2} = \frac{(4-2x)(4+2x)}{(x^2 + 4)^2} = 0 \\ &\Rightarrow (4-2x)(4+2x) = 0 \\ &\Rightarrow x = 2 \quad or \quad x = -2\end{aligned}$$

Use the second order condition to check the property of the critical points:

$$\begin{aligned}\frac{\partial^2 f(x)}{\partial x^2} &= \frac{-8x(x^2 + 4)^2 - 4x(x^2 + 4)(16 - 4x^2)}{(x^2 + 4)^4} \\ For \quad x &= 2, \quad f''(x) = -0.25 < 0 \Rightarrow x = 2 \quad \text{is} \quad \text{a local maximum.} \\ For \quad x &= -2, \quad f''(x) = 0.25 > 0 \Rightarrow x = -2 \quad \text{is} \quad \text{a local minimum.}\end{aligned}$$

d $f(x) = \frac{1}{4}x^4 - \frac{4}{3}x^3 + \frac{1}{2}x^2 + 6x$, One root is $x = -1$.

Set the first order condition equal to 0 to find the critical points:

$$\begin{aligned} \frac{\partial f(x)}{\partial x} &= x^3 - 4x^2 + x + 6 = 0 \\ &\Rightarrow (x-3)(x-2)(x+1) = 0 \\ &\Rightarrow x = 3 \quad or \quad x = 2 \quad or \quad x = -1 \end{aligned}$$

Use the second order condition to check the property of the critical points:

$$\begin{aligned} \frac{\partial^2 f(x)}{\partial x^2} &= 3x^2 - 8x + 1 \\ For \quad x &= 3, \quad f''(x) = 4 > 0 \Rightarrow x = 3 \quad \text{is} \quad \text{a local minimum.} \\ For \quad x &= 2, \quad f''(x) = -3 < 0 \Rightarrow x = 2 \quad \text{is} \quad \text{a local maximum.} \\ For \quad x &= -1, \quad f''(x) = 12 > 0 \Rightarrow x = -1 \quad \text{is} \quad \text{a local minimum.} \end{aligned}$$

e $f(x) = xe^{-x}$

Set the first order condition equal to 0 to find the critical points:

$$\frac{\partial f(x)}{\partial x} = e^{-x} - xe^{-x} = 0$$
$$\Rightarrow e^{-x}(1-x) = 0$$
$$\Rightarrow x = 1$$

Use the second order condition to check the property of the critical points:

$$\frac{\partial^2 f(x)}{\partial x^2} = xe^{-x} - 2e^{-x}$$
$$For \quad x = 1, \quad f''(x) = -e^{-1} < 0 \Rightarrow x = 1 \quad \text{is} \quad \text{a local maximum.}$$

Problem 2. Consider the following matrices.

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 5 & -5 \\ 4 & -4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 & 2 \\ -2 & 5 & -2 \\ 6 & -11 & 7 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ -3 & -4 & -2 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & -1 & 4 \\ 1 & 0 & 2 \\ 4 & -1 & 7 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 4 & 1 \\ 1 & 0 & 2 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 7 \end{bmatrix}$$

and vectors

$$a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Compute the following

a A + B

$$A + B = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 5 & -5 \\ 4 & -4 & 5 \end{bmatrix} + \begin{bmatrix} 1 & -3 & 2 \\ -2 & 5 & -2 \\ 6 & -11 & 7 \end{bmatrix} = \begin{bmatrix} 2 & -5 & 4 \\ -4 & 10 & -7 \\ 10 & -15 & 12 \end{bmatrix}$$

b AB

$$AB = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 5 & -5 \\ 4 & -4 & 5 \end{bmatrix} \times \begin{bmatrix} 1 & -3 & 2 \\ -2 & 5 & -2 \\ 6 & -11 & 7 \end{bmatrix} = \begin{bmatrix} 17 & -35 & 20 \\ -42 & 86 & -49 \\ 42 & -87 & 51 \end{bmatrix}$$

c c'B

$$\begin{aligned} c'B &= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}' \times \begin{bmatrix} 1 & -3 & 2 \\ -2 & 5 & -2 \\ 6 & -11 & 7 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & -3 & 2 \\ -2 & 5 & -2 \\ 6 & -11 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -8 & 4 \end{bmatrix} \end{aligned}$$

d a'F

$$\begin{aligned} a'F &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}' \times \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 10 & 15 \end{bmatrix} \end{aligned}$$

e FA

$$FA = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 7 \end{bmatrix} \times \begin{bmatrix} 1 & -2 & 2 \\ -2 & 5 & -5 \\ 4 & -4 & 5 \end{bmatrix} = \begin{bmatrix} 9 & -4 & 7 \\ 16 & -9 & 14 \\ 21 & -9 & 16 \end{bmatrix}$$

f EA

$$EA = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 4 & 1 \\ 1 & 0 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & -2 & 2 \\ -2 & 5 & -5 \\ 4 & -4 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 2 \\ -4 & 16 & -15 \\ 9 & -10 & 12 \end{bmatrix}$$

g Fb

$$Fb = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 7 \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} -7 \\ -11 \\ -15 \end{bmatrix}$$

h C+E

$$C+E=\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ -3 & -4 & -2 \end{bmatrix}+\begin{bmatrix} 1 & 2 & 2 \\ 0 & 4 & 1 \\ 1 & 0 & 2 \end{bmatrix}=\begin{bmatrix} 2 & 4 & 3 \\ 2 & 9 & 3 \\ -2 & -4 & 0 \end{bmatrix}$$

i C’

$$C'=\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ -3 & -4 & -2 \end{bmatrix}'=\begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -4 \\ 1 & 2 & -2 \end{bmatrix}$$

Problem 3. Consider the following matrix. Use elementary row operations on the matrix to reduce columns 1-3 to an identity matrix.

$$G = \begin{bmatrix} 1 & -3 & 2 & 1 \\ -2 & 5 & -2 & 2 \\ 4 & -11 & 7 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 2 & 1 \\ -2 & 5 & -2 & 2 \\ 4 & -11 & 7 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 2 & 1 \\ 0 & -1 & 2 & 4 \\ 0 & 1 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 2 & 1 \\ 0 & -1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 2 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$