

**ECONOMICS 207**  
**SPRING 2007**  
**FINAL EXAMINATION**

For your information, the bordered Hessian in the constrained optimization problem written as

$$\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda g(x_1, x_2)$$

is given by

$$H_B = \begin{bmatrix} \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1 \partial x_1} & \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1 \partial x_2} & \frac{\partial g(x_1, x_2)}{\partial x_1} \\ \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2 \partial x_1} & \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2 \partial x_2} & \frac{\partial g(x_1, x_2)}{\partial x_2} \\ \frac{\partial g(x_1, x_2)}{\partial x_1} & \frac{\partial g(x_1, x_2)}{\partial x_2} & 0 \end{bmatrix}$$

**Problem 1.** [40 points]

Below you are given a production function for a competitive firm. You are also given the price of the firm's output and the prices of the two inputs used by the firm. Output price is represented by  $p$ , the price of the first input by  $w_1$  and the price of the second input by  $w_2$ .

Hint:  $560 - 144 = 416$ .

$$f(x_1, x_2) = 40x_1 + 20x_2 - 2x_1^2 + x_1x_2 - x_2^2$$

$$p = 10$$

$$w_1 = 120, \quad w_2 = 60$$

- a. Write an equation that represents profit as a function of the two inputs  $x_1$  and  $x_2$ . Simplify the expression.
- b. Find all first and second partial derivatives of the function.

$\frac{\partial \pi}{\partial x_1}$	$\frac{\partial \pi}{\partial x_2}$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2}$

c. Find potential profit maximizing levels of  $x_1$  and  $x_2$ .

- d. Fill in the elements of the Hessian matrix of the profit equation evaluated at the critical values of  $x_1$  and  $x_2$  and then verify the optimal levels of  $x_1$  and  $x_2$ .

$$\begin{array}{|l} \frac{\partial^2 \pi}{\partial x_1 \partial x_1} = \\ \\ \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} = \end{array} \begin{array}{|l} \frac{\partial^2 \pi}{\partial x_1 \partial x_2} = \\ \\ \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_2} = \end{array} \begin{array}{|l} \\ \\ \\ = \end{array}$$

e. What is the optimal level of output?

f. How much does the firm spend on inputs?

**Problem 2.** [40 points]

Consider a consumer with a utility function given by

$$v = u(x_1, x_2) = 40x_1 + 20x_2 - 2x_1^2 + x_1x_2 - x_2^2$$

The consumer faces prices and an income constraint given by

$$p_1 = 120, \quad p_2 = 60, \quad m_0 = 1920$$

Find potential levels of  $x_1$  and  $x_2$  to maximum utility for this consumer given the income constraint and the stated prices. Verify that these consumption levels maximize utility.

Hint:  $1920/160=12$ ;  $4 \times 6 \times 6 = 144$ .

- a. Set up the objective function for this problem and find all first and second partial derivatives of the function with respect to  $x_1$  and  $x_2$ .

$$\mathcal{L}(x_1, x_2, \lambda) =$$

$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1}$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2}$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2}$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2}$

b. What is the derivative of the objective function in this problem with respect to  $\lambda$ ?

c. Find the partial derivatives of the constraint equation with respect to  $x_1$  and  $x_2$ .

$\frac{\partial g(x_1, x_2)}{\partial x_1}$	$\frac{\partial g(x_1, x_2)}{\partial x_2}$
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d. Use the information from 2a and 2b to find critical values for  $x_1$ ,  $x_2$  and  $\lambda$ .



- e. Use the answers from part 2d and the expressions from parts 2a and 2c to fill in the bordered Hessian matrix for this problem. Then determine whether the critical values indicate a maximum or a minimum. The determinant of the bordered Hessian matrix is 57,600.

$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} =$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} =$	$\frac{\partial g(x_1, x_2)}{\partial x_1} =$	=
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} =$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} =$	$\frac{\partial g(x_1, x_2)}{\partial x_2} =$	
$\frac{\partial g(x_1, x_2)}{\partial x_1} =$	$\frac{\partial g(x_1, x_2)}{\partial x_2} =$	0	

**Problem 3.** [40 points]

Below you are given a production function for a competitive firm. You are also given the price of the firm's output and the prices of the two inputs used by the firm. Output price is represented by  $p$ , the price of the first input by  $w_1$  and the price of the second input by  $w_2$ .

$$f(x_1, x_2) = x_1^{1/5} x_2^{1/2}$$

$$p = 80$$

$$w_1 = 5, \quad w_2 = 16$$

a. Write an equation that represents profit as a function of the two inputs  $x_1$  and  $x_2$ . Simplify the expression.

b. Find all first and second partial derivatives of the function.

$\frac{\partial \pi}{\partial x_1}$	$\frac{\partial \pi}{\partial x_2}$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2}$

c. Find potential profit maximizing levels of  $x_1$  and  $x_2$ .

- d. Fill in the elements of the Hessian matrix of the profit equation evaluated at the critical values of  $x_1$  and  $x_2$  and then verify the optimal levels of  $x_1$  and  $x_2$ .

$$\begin{array}{|c|} \hline \frac{\partial^2 \pi}{\partial x_1 \partial x_1} = \\ \\ \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} = \\ \hline \end{array}
 \quad
 \begin{array}{|c|} \hline \frac{\partial^2 \pi}{\partial x_1 \partial x_2} = \\ \\ \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -\frac{8}{25} \\ \hline \end{array}
 \quad
 \begin{array}{|c|} \hline \\ \\ \\ = \\ \hline \end{array}$$

The discriminant is  $\frac{3}{100}$ .

e. What is the optimal level of output?

f. How much does the firm spend on inputs?

**Problem 4.** [40 points]

Consider a firm with a production function given by

$$f(x_1, x_2) = x_1^{1/5} x_2^{1/2}$$

The firm faces prices and a cost constraint given by

$$w_1 = 5$$

$$w_2 = 16$$

$$c_0 = 560$$

Find potential levels of  $x_1$ ,  $x_2$  and  $\lambda$  to maximize output for this firm given the cost constraint and the stated prices. Verify that these input levels maximize output.

- a. Set up the objective function for this problem and find all first and second partial derivatives of the function with respect to  $x_1$  and  $x_2$ .

$$\mathcal{L}(x_1, x_2, \lambda) =$$

$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1}$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2}$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2}$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2}$

- b. What is the derivative of the objective function in this problem with respect to  $\lambda$ ?

- c. Find the partial derivatives of the constraint equation with respect to  $x_1$  and  $x_2$ .

$\frac{\partial g(x_1, x_2)}{\partial x_1} =$	$\frac{\partial g(x_1, x_2)}{\partial x_2} =$
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- d. Use the information from 4a and 4b to find critical values for  $x_1$ ,  $x_2$  and  $\lambda$ .



- e. Use the answers from part 4d and the expressions from parts 4a and 4c to fill in the bordered Hessian matrix for this problem. Then determine whether the critical values indicate a maximum or a minimum. The determinant of the bordered Hessian is  $\frac{7}{10}$ .

Hints:  $\frac{256}{640} = \frac{3}{10}$

$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} =$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = \frac{1}{800}$	$\frac{\partial g(x_1, x_2)}{\partial x_1} = 5$	$=$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} =$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} =$	$\frac{\partial g(x_1, x_2)}{\partial x_2} =$	
$\frac{\partial g(x_1, x_2)}{\partial x_1} =$	$\frac{\partial g(x_1, x_2)}{\partial x_2} =$	$0$	

f. How much output can this firm produce given it spends only \$560?

**Problem 5.** [40 points]

Consider a producer with a production function given by

$$f(x_1, x_2) = x_1^{1/5} x_2^{1/2}$$

The firm faces prices and an output target given by

$$w_1 = 5, \quad w_2 = 16, \quad y_0 = 10$$

Find potential levels of  $x_1$  and  $x_2$  to minimize the cost for this producer to reach the target level of output given the stated prices. Verify that these input levels minimize cost.

- a. Set up the objective function for this problem and find all first and second partial derivatives of the function with respect to  $x_1$  and  $x_2$ .

$$\mathcal{L}(x_1, x_2, \lambda) =$$

$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1}$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2}$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2}$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2}$

b. What is the derivative of the objective function in this problem with respect to  $\lambda$ ?

c. Find the partial derivatives of the constraint equation with respect to  $x_1$  and  $x_2$ .

$\frac{\partial g(x_1, x_2)}{\partial x_1}$	$\frac{\partial g(x_1, x_2)}{\partial x_2}$
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d. Use the information from 5a and 5b to find critical values for  $x_1$ ,  $x_2$  and  $\lambda$ .

e. Substitute the first set of values for  $x_1$ ,  $x_2$  and  $\lambda$  into the bordered Hessian matrix. Show that the determinant of this matrix is  $-\frac{7}{800}$ .

Hints:  $16 \times 25 \times 16 = 6400$ ;  $\lambda = 80$ .

$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = \frac{1}{8}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} =$	$\frac{\partial g(x_1, x_2)}{\partial x_1} = \frac{1}{16}$	$=$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} =$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} =$	$\frac{\partial g(x_1, x_2)}{\partial x_2} =$	
$\frac{\partial g(x_1, x_2)}{\partial x_1} =$	$\frac{\partial g(x_1, x_2)}{\partial x_2} =$	$0$	

f. How much does this firm spend on inputs?

g. How much output does it produce?

h. Why did you not need to do any of the work in problem 5 given you had already worked problems 3 and 4?