ECONOMICS 207
SPRING 2007
LABORATORY EXERCISE 13

Problem 1. For the following problem, write an equation that represents profit as a function of the input x. Write it in the form π = pf(x)-wx and then simplify the expression. Then find the first and second derivatives of the function. Then find the critical points. For each critical point state whether profit is at a relative maximum, relative minimum, or otherwise. Check to see if there are points of inflection **at points other than** critical points.

$$\begin{aligned} f(x) &= 30x \ + \ 30x^2 \ - \ 2x^3 \\ p &= 2 \\ w &= 168 \end{aligned}$$

The profit function:

$$\begin{aligned} \pi =& pf(x) - wx \\ =& 2(30x + 30x^2 - 2x^3) - 168x \\ =& 60x + 60x^2 - 4x^3 - 168x \\ =& -4x^3 + 60x^2 - 108x \end{aligned}$$

To find the critical points, we first find the first and second derivatives.

$$\begin{aligned} \pi' =& -12x^2 + 120x - 108 \\ \pi'' =& -24x + 120 \end{aligned}$$

We then set the first derivative to 0.

$$\begin{aligned} \pi' =& 0 \\ \rightarrow -12x^2 + 120x - 108 =& 0 \\ x^2 - 10x + 9 =& 0 \\ (x-9)(x-1) =& 0 \\ x =& 1, 9. \end{aligned}$$

Hence, the critical points are 1 and 9. To check which one is the optimal one, we plug them into the second derivative.

$$\begin{aligned} \pi''(1) =& -24 + 120 \\ =& 96 > 0 \\ \pi''(9) =& -24 \cdot 9 + 120 \\ =& -96 < 0 \end{aligned}$$

Thus, 9 is the optimal level of input.

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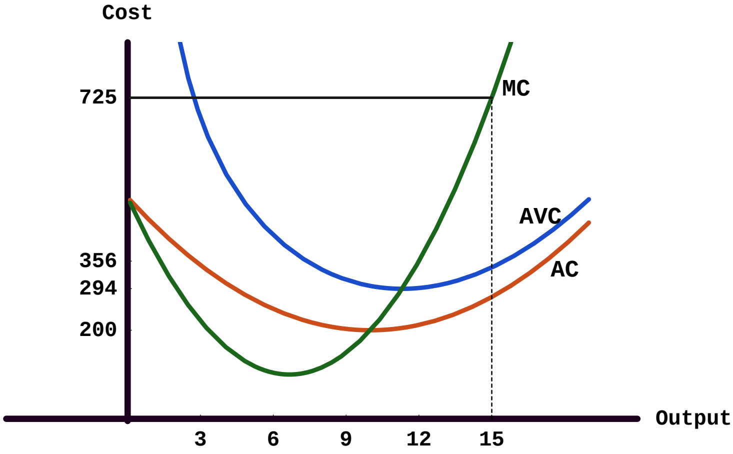
To find inflection point, we let second derivative to be 0.

$$\begin{aligned} \pi'' =& 0 \\ \rightarrow 24x =& 120 \\ x =& 5 \end{aligned}$$

So, there exists one inflection point of 5 other than critical points.

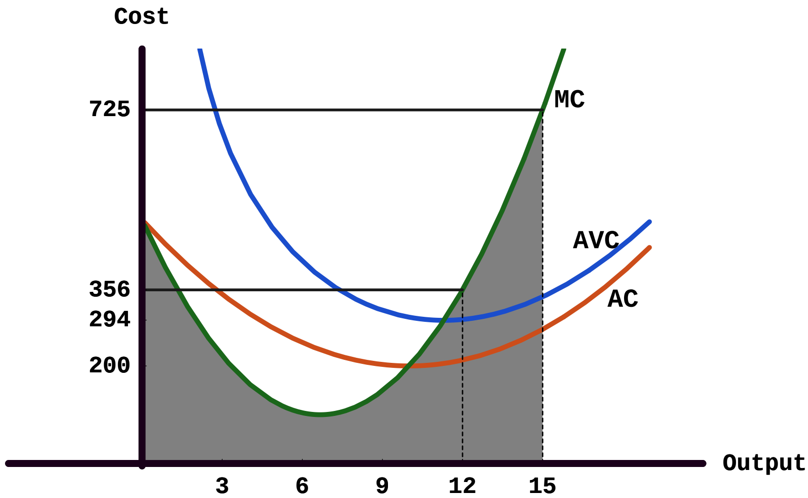
Problem 2. The cost function for a firm is a rule or mapping that tells the total cost of production of any output level produced by the firm. If the variable y represents the output of the firm, then the cost function is given by c(y). Marginal cost represents the change in the cost of production for the firm as output changes and is given by the derivative of the cost function with respect to output,i.e., Marginal Cost(MC) = $\frac{dc(y)}{dy}$. A competitive firm facing a fixed output price maximizes profit at the output level where marginal cost is equal to price as in the figure 1.

FIGURE 1. Profit Maximization



The area below the cost curve is a measure of variable cost and can be found by integrating the marginal cost curve from 0 to any given output level y. The shaded area in figure 2 represents the variable cost of production for the cost function $c(y) = 1000 + 500y - 60\,y^2 + 3y^3$.

FIGURE 2. Variable Cost of Production and Producer Surplus



Producer surplus is the area below a given price and above the marginal cost curve. Producer surplus is the unshaded area below the horizontal line at 725 in figure 2. Producer surplus can be computed by subtracting the shaded area from total revenue.

a. Find the profit maximizing level of output for the following firm. Demonstrate that the level you choose maximizes profit.

$$price \;=\; p \;= \$725$$

$$cost \;=\; c(y) \;= 1000 \;+\; 500y \;-\; 60y^2 \;+\; 3y^3$$

$$\begin{aligned} \pi =& py - cy \\ =& 725y - 1000 - 500y + 60y^2 - 3y^3 \\ =& -1000 + 225y + 60y^2 - 3y^3 \end{aligned}$$

The first and second derivatives are

$$\begin{aligned} \pi' =& 225 + 120y - 9y^2 \\ \pi'' =& 120 - 18y \end{aligned}$$

To find the optimal level, we first set the first derivative to 0.

$$\begin{aligned} \pi' =& 0 \\ \rightarrow 225 + 120y - 9y^2 =& 0 \\ 75 + 40y - 3y^2 =& 0 \\ 3y^2 - 40y - 75 =& 0 \\ (3y + 5)(y - 15) =& 0 \\ y =& -\frac{5}{3}, 15. \\ \pi''(-\frac{5}{3}) =& 120 + 18 \cdot \frac{5}{3} \\ =& 120 + 30 \\ =& 150 > 0 \\ \pi''(15) =& 120 - 18 \cdot 15 \\ =& 120 - 270 \\ =& -150 < 0 \end{aligned}$$

Thus, 15 maximizes the firm's profit.

b. Find the profit maximizing level of output when the price is \$356. Demonstrate that the level you choose maximizes profit.

The profit is

$$\begin{aligned}\pi =& 356y - 1000 - 500y + 60y^2 - 3y^3\\ =& -1000 - 144y + 60y^2 - 3y^3.\end{aligned}$$

The first and second derivatives are

$$\begin{aligned}\pi' =& -144 + 120y - 9y^2\\ \pi'' =& 120 - 18y\end{aligned}$$

We set the first derivative to 0.

$$\begin{aligned}\pi' =& 0\\ \rightarrow -144 + 120y - 9y^2 =& 0\\ 3y^2 - 40y + 48 =& 0\\ (y-12)(3y-4) =& 0\\ y =& 12, \frac{4}{3}.\end{aligned}$$

We then plug the critical points into the second derivative.

$$\begin{aligned}\pi''(12) =& 120 - 18 \cdot 12\\ =& 120 - 216\\ =& -96 < 0\\ \pi''(\frac{4}{3}) =& 120 - 18 \cdot \frac{4}{3}\\ =& 120 - 24\\ =& 96 > 0.\end{aligned}$$

Hence, 12 is the optimal level of output.

c. Show that variable cost for this firm is \$4125 when price is \$725?

When price is 725, the optimal output is 15.

$$\begin{aligned}\text{Variable Cost} =& 500y - 60y^2 + 3y^3\\ =& 500 \cdot 15 - 60 \cdot 15^2 + 3 \cdot 15^3\\ =& 4125\end{aligned}$$

d. What is producer surplus for this firm when the price is \$725?

$$\begin{aligned}\text{Producer Surplus} =& \int_0^{15} (p - MC(y))dy\\ =& \int_0^{15} (725 - 500 + 120y - 9y^2)dy\\ =& \int_0^{15} (225 + 120y - 9y^2)dy\\ =& (225y + 60y^2 - 3y^3)|_{15}\\ =& 6750\end{aligned}$$

e. What is variable cost for this firm when price is \$356?

When the price is 365, the output level is 12.

$$\begin{aligned}\text{Variable cost} =& 500 \cdot 12 - 60 \cdot 12^2 + 3 \cdot 12^3\\ =& 2544\end{aligned}$$

f. What is producer surplus for this firm when the price is \$356?

$$\begin{aligned}\text{Producer Surplus} =& \int_0^{12} (p - MC(y))dy\\ =& \int_0^{12} (365 - 500 + 120y - 9y^2)dy\\ =& \int_0^{12} (-144 + 120y - 9y^2)dy\\ =& (-144y + 60y^2 - 3y^3)|_{12}\\ =& 1728\end{aligned}$$

g. Show that the firm is $5022 worse off when price falls from $725 to $356?

The producer surplus drops from 6750 to 1728, which is 5022 worse off.

h. Cross-hatch the change in producer surplus in Figure 2.

Problem 3. For each of the following problems, write an equation that represents profit as a function of the two inputs x_1 and x_2. Write it in the form π = pf(x_1,x_2)-w_1x_1-w_2x_2 and then simplify the expression. Then find all first and second partial derivatives of the function.

a.

$$
\begin{aligned}
f(x_1, x_2) &= 10x_1 + 40x_2 - x_1^2 + x_1x_2 - x_2^2 \\
p &= 4 \\
w_1 &= 60, \qquad w_2 = 24
\end{aligned}
$$

$$
\begin{aligned}
\pi =& pf - w_1x_1 - w_2x_2 \\
=& 40x_1 + 160x_2 - 4x_1^2 + 4x_1x_2 - 4x_2^2 - 60x_1 - 24x_2 \\
=& -20x_1 + 136x_2 - 4x_1^2 + 4x_1x_2 - 4x_2^2
\end{aligned}
$$

$\frac{\partial \pi}{\partial x_1} = -20 - 8x_1 + 4x_2$	$\frac{\partial \pi}{\partial x_2} = 136 + 4x_1 - 8x_2$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -8$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 4$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 4$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -8$

Find potential profit maximizing levels of x_1 and x_2.

$$
\begin{aligned}
\frac{\partial \pi}{\partial x_1} =& 0 \\
\frac{\partial \pi}{\partial x_2} =& 0
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
-20 - 8x_1 + 4x_2 =& 0 \\
136 + 4x_1 - 8x_2 =& 0
\end{aligned}
$$

$\rightarrow$

$$
\begin{aligned}
-5 - 2x_1 + x_2 =& 0 \\
34 + x_1 - 2x_2 =& 0
\end{aligned}
$$

Multiply the first equation by 2 and add it to the second one.

$$
\begin{aligned}
24 - 3x_1 =& 0 \\
x_1 =& 8 \\
\text{and } x_2 =& 2x_1 + 5 \\
=& 21
\end{aligned}
$$

By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

The determinant of the Hessian matrix is $64 - 16 = 48$. Since $\frac{\partial^2 \pi}{\partial x_1 \partial x_1} < 0$ and the determinant is positive, $(8, 21)$ is the optimal level.

b.

$$
\begin{aligned}
f(x_1, x_2) &= 45x_1 + 15x_2 - 2x_1^2 + 2x_1x_2 - x_2^2 \\
p &= 10 \\
w_1 &= 150, \qquad w_2 = 10
\end{aligned}
$$

$$
\begin{aligned}
\pi =& 450x_1 + 150x_2 - 20x_1^2 + 20x_1x_2 - 10x_2^2 - 150x_1 - 10x_2 \\
=& 300x_1 + 140x_2 - 20x_1^2 + 20x_1x_2 - 10x_2^2
\end{aligned}
$$

$\frac{\partial \pi}{\partial x_1} = 300 - 40x_1 + 20x_2$	$\frac{\partial \pi}{\partial x_2} = 140 + 20x_1 - 20x_2$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -40$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 20$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 20$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -20$

Find potential profit maximizing levels of x_1 and x_2.

$$\begin{aligned}\frac{\partial \pi}{\partial x_1} &= 0\\ \frac{\partial \pi}{\partial x_2} &= 0\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}300 - 40x_1 + 20x_2 &= 0\\ 140 + 20x_1 - 20x_2 &= 0\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}15 - 2x_1 + x_2 &= 0\\ 7 + x_1 - x_2 &= 0\\ \rightarrow 22 - x_1 &= 0\\ x_1 &= 22\\ x_2 = x_1 + 7\\ &= 29\end{aligned}$$

By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

The determinant is $800 - 400 = 400 > 0$. Since it is positive, $(22, 29)$ is the optimal one.

c.

$$\begin{aligned} f(x_1, x_2) &= x_1^{1/2} x_2^{1/3} \\ p &= 594 \\ w_1 &= 81, \qquad w_2 \ = \ 242 \end{aligned}$$

$$\pi = 594 x_1^{1/2} x_2^{1/3} - 81 x_1 - 242 x_2$$

$\frac{\partial \pi}{\partial x_1} = 297 x_1^{-1/2} x_2^{1/3} - 81$	$\frac{\partial \pi}{\partial x_2} = 198 x_1^{1/2} x_2^{-2/3} - 242$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -\frac{297}{2} x_1^{-3/2} x_2^{1/3}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 99 x_1^{-1/2} x_2^{-2/3}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 99 x_1^{-1/2} x_2^{-2/3}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -132 x_1^{1/2} x_2^{-5/3}$

Find potential profit maximizing levels of x_1 and x_2.

$$297x_1^{-1/2}x_2^{1/3} - 81 = 0$$
$$198x_1^{1/2}x_2^{-2/3} - 242 = 0$$

$\Rightarrow$

$$\frac{297x_2}{198x_1} = \frac{81}{242}$$
$$\rightarrow x_2 = \frac{27}{121}x_1$$
$$\rightarrow 297x_1^{-1/2}(\frac{27}{121}x_1)^{1/3} = 81$$
$$\rightarrow 297x_1^{-1/2+1/3}3(11)^{-2/3} = 81$$
$$\rightarrow 27 \cdot 11x_1^{-1/6}(11)^{-2/3} = 27$$
$$\rightarrow 11^{1-2/3} = x_1^{1/6}$$
$$\rightarrow 11^{1/3} = x_1^{1/6}$$
$$\rightarrow x_1 = 11^2 = 121$$
$$\text{and } x_2 = 27$$

By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

$$\text{Det } = \begin{vmatrix} -\frac{81}{242} & 1 \\ 1 & -\frac{1452}{243} \end{vmatrix} = \frac{81}{242} \cdot \frac{1452}{243} - 1 = 2 - 1 = 1 \tag{1}$$

Since the determinant is positive, $(121, 27)$ is the optimal output level.

d.

$$\begin{aligned} f(x_1, x_2) &= x_1^{1/3} x_2^{2/7} \\ p &= 84 \\ w_1 &= 7, \qquad w_2 \;=\; 3 \end{aligned}$$

The profit function is

$$\pi = 84x_1^{1/3} x_2^{2/7} - 7x_1 - 3x_2$$

$\frac{\partial \pi}{\partial x_1} = 28x_1^{-2/3} x_2^{2/7} - 7$	$\frac{\partial \pi}{\partial x_2} = 24x_1^{1/3} x_2^{-5/7} - 3$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -\frac{56}{3} x_1^{-5/3} x_2^{2/7}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 8x_1^{-2/3} x_2^{-5/7}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 8x_1^{-2/3} x_2^{-5/7}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -\frac{120}{7} x_1^{1/3} x_2^{-12/7}$

Find potential profit maximizing levels of x_1 and x_2.

We set the first derivatives to 0.

$$\begin{aligned} 28x_1^{-2/3}x_2^{2/7} &= 7 \\ 24x_1^{1/3}x_2^{-5/7} &= 3 \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} 4x_1^{-2/3}x_2^{2/7} &= 1 \\ 8x_1^{1/3}x_2^{-5/7} &= 1 \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} \frac{1x_2}{2x_1} &= 1 \\ x_2 &= 2x_1 \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} 4x_1^{-2/3}(2x_1)^{2/7} &= 1 \\ 2^{2+2/7}x_1^{-2/3+2/7} &= 1 \\ 2^{16/7}x_1^{-8/21} &= 1 \\ x_1^{8/21} &= 2^{16/7} \\ x_1 &= 2^6 = 64 x_2 = \quad 128 \end{aligned}$$

By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

$$\text{Det}\begin{vmatrix} -\frac{7}{96} & \frac{1}{64} \\ \frac{1}{64} & -\frac{15}{896} \end{vmatrix} = \frac{7}{96} \cdot \frac{15}{896} - \left(\frac{1}{64}\right)^2 = \frac{1}{1024} \tag{2}$$

Since the determinant is positive, $(64, 128)$ is the optimal level.

Problem 4. Find the listed partial derivatives of each of the following functions.

a. $\mathcal{L}(x_1, x_2, \lambda) \;=\; 60x_1 + 24x_2 - \lambda\left(10x_1 + 40x_2 - x_1^2 + x_1x_2 - x_2^2 - 583\right)$

$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 60 - \lambda(10 - 2x_1 + x_2)$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = 24 - \lambda(40 + x_1 - 2x_2)$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = -(10x_1 + 40x_2 - x_1^2 + x_1x_2 - x_2^2 - 583)$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = 2\lambda$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = -\lambda$	-$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = 10 - 2x_1 + x_2$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = -\lambda$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = 2\lambda$	- $\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = 40 + x_1 - 2x_2$
- $\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = 10 - 2x_1 + x_2$	-$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = 40 + x_1 - 2x_2$	-$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0$

b. $\mathcal{L}(x_1, x_2, \lambda) \;=\; 45x_1 + 15x_2 - 2x_1^2 + 2x_1x_2 - x_2^2 - \lambda\,(150x_1 \;+\; 10x_2 \;-\; 3590)$

$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 45 - 4x_1 + 2x_2 - 150\lambda$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = 15 + 2x_1 - 2x_2 - 1 - \lambda$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = -(150x_1 + 10x_2 - 3590)$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -4$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = 2$	-$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = 150$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = 2$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -2$	- $\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = 10$
- $\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = 150$	-$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = 10$	-$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0$

c. $\mathcal{L}(x_1, x_2, \lambda) \;=\; 81x_1 + 242x_2 - \lambda\left(x_1^{1/2}x_2^{1/3} - 33\right)$

$\frac{\partial \mathcal{L}(x_1,x_2,\lambda)}{\partial x_1} = 81 - \frac{1}{2}x_1^{-1/2}x_2^{1/3}\lambda$	$\frac{\partial \mathcal{L}(x_1,x_2,\lambda)}{\partial x_2} = 242 - \frac{1}{3}x_1^{1/2}x_2^{-2/3}\lambda$	$\frac{\partial \mathcal{L}(x_1,x_2,\lambda)}{\partial \lambda} = x_1^{1/2}x_2^{1/3} - 33$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = \frac{1}{4}\lambda x_2^{1/3}x_1^{-3/2}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = -\frac{\lambda}{6}x_2^{-2/3}x_1^{-1/2}$	$-\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = \frac{1}{2}x_2^{1/3}x_1^{-1/2}$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = -\frac{\lambda}{6}x_2^{-2/3}x_1^{-1/2}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = \frac{2}{9}\lambda x_2^{-5/3}x_1^{1/2}$	$-\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = \frac{1}{3}x_2^{-2/3}x_1^{1/2}$
$-\ \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = \frac{1}{2}x_2^{1/3}x_1^{-1/2}$	$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = \frac{1}{3}x_2^{-2/3}x_1^{1/2}$	$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0$

d. $\mathcal{L}(x_1, x_2, \lambda) \;=\; x_1^{1/3} x_2^{2/7} - \lambda\,(7x_1 + 3x_2 - 832)$

$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = \frac{1}{3} x_1^{-2/3} x_2^{2/7} - 7\lambda$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = \frac{2}{7} x_1^{1/3} x_2^{-5/7} - 3\lambda$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = 7x_1 + 3x_2 - 832$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -\dfrac{2x_2^{2/7}}{9x_1^{5/3}}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = \dfrac{2}{21} x_1^{-2/3} x_2^{-5/7}$	-$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = 7$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = \dfrac{2}{21} x_1^{-2/3} x_2^{-5/7}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -\frac{10}{49} x_1^{1/3} x_2^{-12/7}$	- $\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = 3$
- $\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = 7$	-$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = 3$	-$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0$

Consider the following matrix and vector.

$$V = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 6 \\ -2 & -3 & -3 \end{bmatrix},$$

$$v = \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix},$$

Problem 5.

a. Find the determinant of the matrix V.

$$\begin{aligned} \text{Det} =& 1 \cdot 4 \cdot (-3) + 1 \cdot 6 \cdot (-2) + 2 \cdot 3 \cdot (-3) \\ & - 2 \cdot 4 \cdot (-2) - 1 \cdot 3 \cdot (-3) - 1 \cdot (-3) \cdot 6 \\ =& - 12 - 12 - 18 + 16 + 9 + 18 \\ =& - 24 + 25 \\ =& 1 \end{aligned}$$

b. Find the inverse of the matrix V using the adjoint method.

$$\text{Cofactor (V)} = \begin{vmatrix} (-12+18) & -(-9+12) & -9+8 \\ -(-3+6) & -3+4 & -(-3+2) \\ 6-8 & -(6-6) & 4-3 \end{vmatrix} = \begin{vmatrix} 6 & -3 & -1 \\ -3 & 1 & 1 \\ -2 & 0 & 1 \end{vmatrix}$$

And

$$\text{Adjoint (V)} = \begin{vmatrix} 6 & -3 & -2 \\ -3 & 1 & 0 \\ -1 & 1 & 1 \end{vmatrix}$$

Since the determinant is 1, the inverse of V is

$$\begin{vmatrix} 6 & -3 & -2 \\ -3 & 1 & 0 \\ -1 & 1 & 1 \end{vmatrix}$$

c. Using the inverse from part b, solve the system of equations

$$V \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = v$$

$$V = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 6 \\ -2 & -3 & -3 \end{bmatrix}, \quad v = \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}$$

Using the inverse of V, we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = V^{-1}v$$

$$= \begin{pmatrix} 6 & -3 & -2 \\ -3 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

d. Using Cramer's rule, solve the system of equations

$$V \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = v$$

$$V = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 6 \\ -2 & -3 & -3 \end{bmatrix}, \quad v = \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}$$

$$\begin{aligned} x_1 =& \frac{Det \begin{pmatrix} -1 & 1 & 2 \\ -2 & 4 & 6 \\ -1 & -3 & -3 \end{pmatrix}}{Det(V)} \\ =& 12 - 6 + 12 + 8 - 6 - 18 \\ =& 32 - 30 \\ =& 2 \\ x_2 =& \frac{Det \begin{pmatrix} 1 & -1 & -2 \\ 3 & -2 & 6 \\ -2 & -1 & -3 \end{pmatrix}}{Det(V)} \\ =& 6 + 12 - 6 - 8 - 9 + 6 \\ =& 10 - 9 \\ =& 1 \\ x_3 =& \frac{Det \begin{pmatrix} 1 & 1 & -1 \\ 3 & 4 & -2 \\ -2 & -3 & -1 \end{pmatrix}}{Det(V)} \\ =& -4 + 4 + 9 - 8 + 3 - 6 \\ =& 1 - 3 \\ =& -2 \end{aligned}$$

e. Using row reduction, find the inverse of the matrix V and the solution to the system of equations

$$V \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = v$$

$$V = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 6 \\ -2 & -3 & -3 \end{bmatrix}, \quad v = \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}$$

We expand the matrix.

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & -1 \\ 3 & 4 & 6 & 0 & 1 & 0 & -2 \\ -2 & -3 & -3 & 0 & 0 & 1 & -1 \end{bmatrix}$$

We first multiply the first row by (-3) and add it to the second one.

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 & 1 & 0 & 1 \\ -2 & -3 & -3 & 0 & 0 & 1 & -1 \end{bmatrix}$$

We then multiply the first one by 2 and add it to the last one.

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 & 1 & 0 & 1 \\ 0 & -1 & 1 & 2 & 0 & 1 & -3 \end{bmatrix}$$

We then add the second row to the last one.

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 & -2 \end{bmatrix}$$

Finally, we subtract the second one from the first row,

$$\begin{bmatrix} 1 & 0 & 2 & 4 & -1 & 0 & -2 \\ 0 & 1 & 0 & -3 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 & -2 \end{bmatrix}$$

and subtract the first row by the last one multiplied by 2.

$$\begin{bmatrix} 1 & 0 & 0 & 6 & -3 & -2 & 2 \\ 0 & 1 & 0 & -3 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 & -2 \end{bmatrix}$$