ECONOMICS 207
SPRING 2007
LABORATORY EXERCISE 14

For your information, the Hessian matrix in the profit maximization problem written as

$$\pi(x_1,\, x_2) \;=\; pf(x_1,\, x_2) \;- w_1x_1 - w_2x_2$$

is given by

$$H(\pi(x_1,\, x_2)) = \begin{bmatrix} \frac{\partial^2 \pi(x_1, x_2)}{\partial x_1 \partial x_1} & \frac{\partial^2 \pi(x_1, x_2)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \pi(x_1, x_2)}{\partial x_2 \partial x_1} & \frac{\partial^2 \pi(x_1, x_2)}{\partial x_2 \partial x_2} \end{bmatrix}$$

The bordered Hessian in the constrained optimization problem written as

$$\mathcal{L}(x_1, x_2, \lambda) \;=\; f(x_1, x_2) - \lambda g(x_1, x_2)$$

is given by

$$H_B = \begin{bmatrix} \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1 \partial x_1} & \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1 \partial x_2} & -\frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2 \partial x_1} & \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2 \partial x_2} & -\frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2 \partial \lambda} \\ -\frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda \partial x_1} & -\frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda \partial x_2} & 0 \end{bmatrix}$$

Date: 16 January 2008.

Problem 1. Given the data below, write an equation that represents profit as a function of the two inputs x_1 and x_2. Write it in the form $\pi = pf(x_1,x_2) - w_1x_1 - w_2x_2$ and then simplify the expression. Then find all first and second partial derivatives of the function.

a.

$$f(x_1, x_2) = 100x_1 + 40x_2 - 2x_1^2 + 2x_1x_2 - 2x_2^2$$
$$p = 1$$
$$w_1 = 16, \qquad w_2 = 40$$

$\pi =$

$\frac{\partial \pi}{\partial x_1}$	$\frac{\partial \pi}{\partial x_2}$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2}$

Find potential profit maximizing levels of x_1 and x_2.

By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

$$\begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1 \partial x_1} = & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} = \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} = & \frac{\partial^2 \pi}{\partial x_2 \partial x_2} = \end{vmatrix} =$$

Problem 2. a. Given the data below, write an equation that represents profit as a function of the two inputs x_1 and x_2. Write it in the form $\pi = pf(x_1,x_2) - w_1x_1 - w_2x_2$ and then simplify the expression. Then find all first and second partial derivatives of the function.

$$f(x_1, x_2) = x_1^{1/6} x_2^{1/3}$$

$$p = 1728$$

$$w_1 = 27, \qquad w_2 = 128$$

$\pi =$

$\frac{\partial \pi}{\partial x_1} =$	$\frac{\partial \pi}{\partial x_2} =$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} =$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} =$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} =$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} =$

b. Show that the profit maximizing levels of x_1 and x_2 are 64 and 27.

c. By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

$$\begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1 \partial x_1} = & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} = \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} = & \frac{\partial^2 \pi}{\partial x_2 \partial x_2} = \end{vmatrix} =$$

Problem 3. a. Find the listed partial derivatives of following function.

$$\mathfrak{L}(x_1, x_2, \lambda) \;=\; 45x_1 + 15x_2 - 2x_1^2 + 2x_1x_2 - x_2^2 - \lambda\,(150x_1 \;+\; 10x_2 \;-\; 3590)$$

$\frac{\partial \mathfrak{L}(x_1, x_2, \lambda)}{\partial x_1} = 45 - 4x_1 + 2x_2 - 150\lambda$	$\frac{\partial \mathfrak{L}(x_1, x_2, \lambda)}{\partial x_2} = 15 + 2x_1 - 2x_2 - 10\lambda$	$\frac{\partial \mathfrak{L}(x_1, x_2, \lambda)}{\partial \lambda} = -150x_1 - 10x_2 + 3590$
$\frac{\partial^2 \mathfrak{L}}{\partial x_1 \partial x_1} = -4$	$\frac{\partial^2 \mathfrak{L}}{\partial x_1 \partial x_2}$	- $\frac{\partial^2 \mathfrak{L}}{\partial x_1 \partial \lambda}$
$\frac{\partial^2 \mathfrak{L}}{\partial x_2 \partial x_1} = 2$	$\frac{\partial^2 \mathfrak{L}}{\partial x_2 \partial x_2}$	- $\frac{\partial^2 \mathfrak{L}}{\partial x_2 \partial \lambda}$
- $\frac{\partial^2 \mathfrak{L}}{\partial \lambda \partial x_1} = 150$	- $\frac{\partial^2 \mathfrak{L}}{\partial \lambda \partial x_2}$	$\frac{\partial^2 \mathfrak{L}}{\partial \lambda \partial \lambda} = 0$

b. Show that the three critical values of the function $\mathcal{L}(x_1, x_2, \lambda)$ are $x_1 = 22$, $x_2 = 29$, and $\lambda = \frac{1}{10}$.

c. Substitute the appropriate values of are x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is 51,400.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -4 & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = 2 & -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = \\ -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = & -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = \end{vmatrix} =$$

A positive determinant indicates a maximum, a negative determinant indicates a minimum.

Problem 4. a. Find the listed partial derivatives of following function.

$$\mathcal{L}(x_1, x_2, \lambda) \;=\; x_1^{1/3} x_2^{2/7} - \lambda\,(7x_1 + 3x_2 - 832)$$

$\frac{\partial \mathcal{L}(x_1,x_2,\lambda)}{\partial x_1} = \frac{1}{3} x_1^{-2/3} x_2^{2/7} - 7\lambda$	$\frac{\partial \mathcal{L}(x_1,x_2,\lambda)}{\partial x_2} = \frac{2}{7} x_1^{1/3} x_2^{-5/7} - 3\lambda$	$\frac{\partial \mathcal{L}(x_1,x_2,\lambda)}{\partial \lambda}$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -\frac{2x_2^{2/7}}{9x_1^{5/3}}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = \frac{2}{21} x_1^{-2/3} x_2^{-5/7}$	$-\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = 7$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -\frac{10}{49} x_1^{1/3} x_2^{-12/7}$	$-\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda}$
$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1}$	$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2}$	$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda}$

b. Show that the three critical values of the function $\mathcal{L}(x_1, x_2, \lambda)$ are $x_1 = 64$, $x_2 = 128$, and $\lambda = \frac{1}{84}$.

c. Substitute the appropriate values of are x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is $\frac{13}{512}$.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -\frac{1}{1152} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = \frac{1}{5376} & -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = \\ -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = & -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = \end{vmatrix} =$$

A positive determinant indicates a maximum, a negative determinant indicates a minimum.

Problem 5. a. Find the listed partial derivatives of following function.

$$\mathcal{L}(x_1, x_2, \lambda) \;=\; 81x_1 + 242x_2 - \lambda\left(x_1^{1/2}x_2^{1/3} - 33\right)$$

$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 81 - \frac{1}{2}\lambda x_1^{-1/2}x_2^{1/3}$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2}$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda}$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = \frac{1}{4}\lambda x_2^{1/3}x_1^{-3/2}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = -\frac{\lambda}{6}x_2^{-2/3}x_1^{-1/2}$	$-\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = \frac{1}{2}x_2^{1/3}x_1^{-1/2}$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} =$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = \frac{2}{9}\lambda x_2^{-5/3}x_1^{1/2}$	$-\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = \frac{1}{3}x_2^{-2/3}x_1^{1/2}$
$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1}$	$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2}$	$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda}$

b. Show that the three critical values of the function $\mathcal{L}(x_1, x_2, \lambda)$ are $x_1 = 121$, $x_2 = 27$, and $\lambda = 594$.

c. Substitute the appropriate values of are x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is $-\frac{5}{18}$.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = \frac{81}{242} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = -1 & -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = \frac{3}{22} \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = \frac{484}{81} & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = \\ -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = & -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = \end{vmatrix} =$$

A positive determinant indicates a maximum, a negative determinant indicates a minimum.

Problem 6. a. Find the listed partial derivatives of following function.

$$\mathcal{L}(x_1, x_2, \lambda) \; = \; 60x_1 + 24x_2 - \lambda\left(10x_1 + 40x_2 - x_1^2 + x_1x_2 - x_2^2 - 583\right)$$

$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 60 - 10\lambda + 2\lambda x_1 - \lambda x_2$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = 24 - 40\lambda - \lambda x_1 + 2\lambda x_2$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = -10x_1 - 40x_2 + x_1^2 - x_1x_2 + x_2^2 + 583$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = 2\lambda$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2}$	$-\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = 10 - 2x_1 + x_2$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = -\lambda$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2}$	$-\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda}$
$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1}$	$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = 40 + x_1 - 2x_2$	$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda}$

b. Show that there are two sets of critical values of the function $\mathcal{L}(x_1, x_2, \lambda)$.

The first set is is $x_1 = 32$, $x_2 = 39$, and $\lambda = -4$.

The second set is $x_1 = 8$, $x_2 = 21$, and $\lambda = 4$.

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The second set of critical values is $x_1 = 8$, $x_2 = 21$, and $\lambda = 4$.

c. Substitute the first set of values for x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is 2,808.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = & -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = \\ -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = & -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = \end{vmatrix} =$$

A positive determinant indicates a maximum, a negative determinant indicates a minimum.

d. Substitute the second set of values for x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is -2,808.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = & -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = \\ -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = & -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = \end{vmatrix} =$$

A positive determinant indicates a maximum, a negative determinant indicates a minimum.