ECONOMICS 207
SPRING 2007
LABORATORY EXERCISE 14

For your information, the Hessian matrix in the profit maximization problem written as

$$\pi(x_1,\, x_2) \;=\; pf(x_1,\, x_2) \;-w_1x_1 - w_2x_2$$

is given by

$$H(\pi(x_1,\, x_2)) = \begin{bmatrix} \dfrac{\partial^2\pi(x_1,x_2)}{\partial x_1\partial x_1} & \dfrac{\partial^2\pi(x_1,x_2)}{\partial x_1\partial x_2} \\ \dfrac{\partial^2\pi(x_1,x_2)}{\partial x_2\partial x_1} & \dfrac{\partial^2\pi(x_1,x_2)}{\partial x_2\partial x_2} \end{bmatrix}$$

The bordered Hessian in the constrained optimization problem written as

$$\mathcal{L}(x_1,x_2,\lambda) \;=\; f(x_1,x_2) - \lambda g(x_1,x_2)$$

is given by

$$H_B = \begin{bmatrix} \dfrac{\partial^2\mathcal{L}(x_1,x_2,\lambda)}{\partial x_1\partial x_1} & \dfrac{\partial^2\mathcal{L}(x_1,x_2,\lambda)}{\partial x_1\partial x_2} & -\dfrac{\partial^2\mathcal{L}(x_1,x_2,\lambda)}{\partial x_1\partial\lambda} \\ \dfrac{\partial^2\mathcal{L}(x_1,x_2,\lambda)}{\partial x_2\partial x_1} & \dfrac{\partial^2\mathcal{L}(x_1,x_2,\lambda)}{\partial x_2\partial x_2} & -\dfrac{\partial^2\mathcal{L}(x_1,x_2,\lambda)}{\partial x_2\partial\lambda} \\ -\dfrac{\partial^2\mathcal{L}(x_1,x_2,\lambda)}{\partial\lambda\partial x_1} & -\dfrac{\partial^2\mathcal{L}(x_1,x_2,\lambda)}{\partial\lambda\partial x_2} & 0 \end{bmatrix}$$

Date: 2 August 2007.

Problem 1. Given the data below, write an equation that represents profit as a function of the two inputs x_1 and x_2. Write it in the form $\pi = pf(x_1,x_2)-w_1x_1-w_2x_2$ and then simplify the expression. Then find all first and second partial derivatives of the function.

a.

$$f(x_1, x_2) = 100x_1 + 40x_2 - 2x_1^2 + 2x_1x_2 - 2x_2^2$$

$$p = 1$$

$$w_1 = 16, \qquad w_2 = 40$$

$$\pi = 100x_1 + 40x_2 - 2x_1^2 + 2x_1x_2 - 2x_2^2 - 16x_1 - 40x_2$$

$$= 84x_1 - 2x_1^2 + 2x_1x_2 - 2x_2^2$$

$\frac{\partial \pi}{\partial x_1} = 84 - 4x_1 + 2x_2$	$\frac{\partial \pi}{\partial x_2} = 2x_1 - 4x_2$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -4$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 2$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 2$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -4$

Find potential profit maximizing levels of x_1 and x_2.

We first set the first derivatives to 0.

$$84 - 4x_1 + 2x_2 = 0$$
$$2x_1 - 4x_2 = 0$$

$\Rightarrow$

$$2x_1 - x_2 = 42$$
$$2x_1 - 4x_2 = 0$$

$\Rightarrow$

$$3x_2 = 42$$
$$\rightarrow x_2 = 14$$

And

$$x_1 = 2x_2$$
$$= 28$$

By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

$$\begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -4 & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 2 \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 2 & \frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -4 \end{vmatrix} = 16 - 4 = 12$$

Problem 2. a. Given the data below, write an equation that represents profit as a function of the two inputs x_1 and x_2. Write it in the form $\pi = pf(x_1,x_2)-w_1x_1-w_2x_2$ and then simplify the expression. Then find all first and second partial derivatives of the function.

$$f(x_1, x_2) = x_1^{1/6} x_2^{1/3}$$
$$p = 1728$$
$$w_1 = 27, \qquad w_2 = 128$$

$$\pi = 1728 x_1^{1/6} x_2^{1/3} - 27x_1 - 128x_2$$

$\frac{\partial \pi}{\partial x_1} = 288 x_1^{-5/6} x_2^{1/3} - 27$	$\frac{\partial \pi}{\partial x_2} = 576 x_1^{1/6} x_2^{-2/3} - 128$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -240 x_1^{-11/6} x_2^{1/3}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 96 x_1^{-5/6} x_2^{-2/3}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 96 x_1^{-5/6} x_2^{-2/3}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -384 x_1^{1/6} x_2^{-5/3}$

b. Show that the profit maximizing levels of x_1 and x_2 are 64 and 27.

Setting the first derivatives to 0, that is,

$$\pi_1 = 0 \tag{1a}$$
$$\pi_2 = 0 \tag{1b}$$

$\Rightarrow$

$$288x_1^{-5/6}x_2^{1/3} = 27$$
$$576x_1^{1/6}x_2^{-2/3} = 128$$

$\Rightarrow$

$$\begin{aligned} \frac{288x_2}{576x_1} &= \frac{27}{128} \\ \rightarrow x_2 &= \frac{27}{64}x_1 \end{aligned} \tag{2}$$

We substitute (2) into the equation (1a), which is

$$\begin{aligned} 288x_1^{-5/6}(\frac{27}{64}x_1)^{1/3} &= 27 \\ \rightarrow 288x_1^{-5/6+1/3}(3)(4^{-1}) &= 27 \\ \rightarrow 8x_1^{-1/2} &= 1 \\ \rightarrow x_1^{1/2} &= 8 \\ \rightarrow x_1 &= 64. \end{aligned} \tag{3}$$

We then plug (3) into (2), that is

$$x_2 = 27.$$

c. By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

$$\begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -\frac{45}{128} & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} = \frac{1}{3} \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} = \frac{1}{3} & \frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -\frac{256}{243} \end{vmatrix} = \frac{45}{128} \cdot \frac{256}{243} - \frac{1}{9} = \frac{7}{27}$$

Problem 3. a. Find the listed partial derivatives of following function.

$$\mathcal{L}(x_1, x_2, \lambda) \;=\; 45x_1 + 15x_2 - 2x_1^2 + 2x_1x_2 - x_2^2 - \lambda\,(150x_1 \;+\; 10x_2 \;-\; 3590)$$

$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 45 - 4x_1 + 2x_2 - 150\lambda$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = 15 + 2x_1 - 2x_2 - 10\lambda$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = -150x_1 - 10x_2 + 3590$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -4$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = 2$	- $\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = 150$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = 2$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -2$	- $\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = 10$
- $\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = 150$	- $\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = 10$	$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0$

b. Show that the three critical values of the function $\mathcal{L}(x_1, x_2, \lambda)$ are $x_1 = 22$, $x_2 = 29$, and $\lambda = \frac{1}{10}$.

Set the first derivatives of profit function to 0.

$$
\begin{aligned}
45 - 4x_1 + 2x_2 &= 150\lambda && \text{(4a)} \\
15 + 2x_1 - 2x_2 &= 10\lambda && \text{(4b)} \\
-150x_1 - 10x_2 + 3590 &= 0 && \text{(4c)}
\end{aligned}
$$

We divide equations (4a) and (4b).

$$
\begin{aligned}
\frac{45 - 4x_1 + 2x_2}{15 + 2x_1 - 2x_2} &= \frac{150\lambda}{10\lambda} = 15 \\
\rightarrow 45 - 4x_1 + 2x_2 &= 225 + 30x_1 - 30x_2 && \text{(5)} \\
\rightarrow 34x_1 - 32x_2 + 180 &= 0
\end{aligned}
$$

We then solve these two equations:

$$
\begin{aligned}
34x_1 - 32x_2 + 180 &= 0, \\
-150x_1 - 10x_2 + 3590 &= 0.
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
17x_1 - 16x_2 + 90 &= 0 \\
15x_1 + x_2 - 359 &= 0
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
17x_1 - 16x_2 + 90 &= 0 \\
240x_1 + 16x_2 - 5744 &= 0
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
257x_1 - 5654 &= 0 \\
\rightarrow x_1 &= 22 \\
\text{and } x_2 &= 359 - 15x_1 \\
\rightarrow &= 359 - 330 \\
&= 29 \\
\text{and } 15 + 2x_1 - 2x_2 &= 10\lambda \\
\rightarrow 10\lambda &= 15 + 2(22 - 29) \\
\rightarrow 10\lambda &= 1 \\
\rightarrow \lambda &= \frac{1}{10}.
\end{aligned}
$$

Substitute the appropriate values of are x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is 51,400.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -4 & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = 2 & -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = 150 \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = 2 & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -2 & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = 10 \\ -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = 150 & -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = 10 & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0 \end{vmatrix} = 2 \cdot 10 \cdot 150 + 150 \cdot 10 \cdot 2 - (-2) \cdot (150)^2 - (-4)10^2 = 51400$$

A positive determinant indicates a maximum, a negative determinant indicates a minimum.

Problem 4. a. Find the listed partial derivatives of following function.

$$\mathcal{L}(x_1, x_2, \lambda) \;=\; x_1^{1/3} x_2^{2/7} - \lambda\,(7x_1 + 3x_2 - 832)$$

$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = \frac{1}{3} x_1^{-2/3} x_2^{2/7} - 7\lambda$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = \frac{2}{7} x_1^{1/3} x_2^{-5/7} - 3\lambda$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = -(7x_1 + 3x_2 - 832)$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -\frac{2x_2^{2/7}}{9x_1^{5/3}}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = \frac{2}{21} x_1^{-2/3} x_2^{-5/7}$	$-\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = 7$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = \frac{2}{21} x_1^{-2/3} x_2^{-5/7}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -\frac{10}{49} x_1^{1/3} x_2^{-12/7}$	$-\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = 3$
$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = 7$	$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = 3$	$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0$

b. Show that the three critical values of the function $\mathcal{L}(x_1, x_2, \lambda)$ are $x_1 = 64$, $x_2 = 128$, and $\lambda = \frac{1}{84}$.

Again, we start from the first derivatives.

$$\frac{1}{3}x_1^{-2/3}x_2^{2/7} = 7\lambda \tag{6a}$$

$$\frac{2}{7}x_1^{1/3}x_2^{-5/7} = 3\lambda \tag{6b}$$

$$7x_1 + 3x_2 - 832 = 0 \tag{6c}$$

$\Rightarrow$

$$\begin{aligned} \frac{7x_2}{6x_1} &= \frac{7}{3} \\ \rightarrow x_2 &= 2x_1 \end{aligned} \tag{7}$$

We plug that into equation (6c), that is

$$\begin{aligned} 7x_1 + 6x_1 &= 832 \\ \rightarrow 13x_1 &= 832 \\ \rightarrow x_1 &= 64 \\ \rightarrow x_2 &= 128 \end{aligned} \tag{8}$$

Finally, we plug x_1, x_2 into (6a).

$$\begin{aligned} 7\lambda &= \frac{1}{3}(4)^{-2}(2)^2 \\ \rightarrow \lambda &= \frac{1}{84} \end{aligned}$$

c. Substitute the appropriate values of are x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is $\frac{13}{512}$.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -\frac{1}{1152} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = \frac{1}{5376} & -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = 7 \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = \frac{1}{5376} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -\frac{5}{25088} & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = 3 \\ -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = 7 & -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = 3 & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0 \end{vmatrix} = 2 \cdot 21 \cdot \frac{1}{5376} + 49 \cdot \frac{5}{25088} + \frac{9}{1152} = \frac{13}{512}$$

A positive determinant indicates a maximum, a negative determinant indicates a minimum.

Problem 5. a. Find the listed partial derivatives of following function.

$$\mathcal{L}(x_1, x_2, \lambda) \;=\; 81x_1 + 242x_2 - \lambda\left(x_1^{1/2}x_2^{1/3} - 33\right)$$

$\dfrac{\partial\mathcal{L}(x_1,x_2,\lambda)}{\partial x_1} = 81 - \frac{1}{2}\lambda x_1^{-1/2}x_2^{1/3}$	$\dfrac{\partial\mathcal{L}(x_1,x_2,\lambda)}{\partial x_2} = 242 - \frac{1}{3}\lambda x_1^{1/2}x_2^{-2/3}$	$\dfrac{\partial\mathcal{L}(x_1,x_2,\lambda)}{\partial \lambda} = -(x_1^{1/2}x_2^{1/3} - 33)$
$\frac{\partial^2\mathcal{L}}{\partial x_1\partial x_1} = \frac{1}{4}\lambda x_2^{1/3}x_1^{-3/2}$	$\frac{\partial^2\mathcal{L}}{\partial x_1\partial x_2} = -\frac{\lambda}{6}x_2^{-2/3}x_1^{-1/2}$	- $\frac{\partial^2\mathcal{L}}{\partial x_1\partial \lambda} = \frac{1}{2}x_2^{1/3}x_1^{-1/2}$
$\frac{\partial^2\mathcal{L}}{\partial x_2\partial x_1} = -\frac{\lambda}{6}x_2^{-2/3}x_1^{-1/2}$	$\frac{\partial^2\mathcal{L}}{\partial x_2\partial x_2} = \frac{2}{9}\lambda x_2^{-5/3}x_1^{1/2}$	- $\frac{\partial^2\mathcal{L}}{\partial x_2\partial \lambda} = \frac{1}{3}x_2^{-2/3}x_1^{1/2}$
- $\frac{\partial^2\mathcal{L}}{\partial \lambda\partial x_1} = \frac{1}{2}x_2^{1/3}x_1^{-1/2}$	- $\frac{\partial^2\mathcal{L}}{\partial \lambda\partial x_2} = \frac{1}{3}x_2^{-2/3}x_1^{1/2}$	$\frac{\partial^2\mathcal{L}}{\partial \lambda\partial \lambda} = 0$

b. Show that the three critical values of the function $\mathcal{L}(x_1, x_2, \lambda)$ are $x_1 = 121$, $x_2 = 27$, and $\lambda = 594$.

We start by setting the first derivatives to 0. That is

$$81 = \frac{1}{2}\lambda x_1^{-1/2} x_2^{1/3} \tag{9a}$$

$$242 = \frac{1}{3}\lambda x_1^{1/2} x_2^{-2/3} \tag{9b}$$

$$x_1^{1/2} x_2^{1/3} - 33 = 0 \tag{9c}$$

$\Rightarrow$

$$\frac{3x_2}{2x_1} = \frac{81}{242} \tag{10}$$
$$\rightarrow x_2 = 27121 x_1$$

We next plug the equation (10) into (9c).

$$\begin{aligned} x_1^{1/2} x_2^{1/3} - 33 &= 0 \\ \rightarrow x_1^{1/2}(27121 x_1)^{1/3} &= 33 \\ \rightarrow x_1^{1/2+1/3}(3)(11)^{-2/3} &= 33 \\ \rightarrow x_1^{1/2+1/3}(11)^{-2/3} &= 11 \\ \rightarrow x_1^{5/6} &= 11^{5/3} \\ \rightarrow x_1 &= 11^2 = 121 \\ \text{and } x_2 &= 27 \end{aligned} \tag{11}$$

Finally, we plug (11) into (9a) and we have.

$$\begin{aligned} 81 &= \frac{1}{2}\lambda x_1^{-1/2} x_2^{1/3} \\ \rightarrow 81 &= \frac{1}{2}\lambda 11^{-1} 3 \\ \rightarrow \lambda &= 594 \end{aligned}$$

c. Substitute the appropriate values of are x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is $-\frac{5}{18}$.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = \frac{81}{242} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = -1 & -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = \frac{3}{22} \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = -1 & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = \frac{484}{81} & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = \frac{11}{27} \\ -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = \frac{3}{22} & -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = \frac{11}{27} & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0 \end{vmatrix}$$

$$\begin{aligned} &=(-1)\cdot\frac{11}{27}\frac{3}{22}+(-1)\cdot\frac{11}{27}\frac{3}{22}-(\frac{3}{22})^2\frac{484}{81}-\frac{81}{242}(\frac{11}{27})^2 \\ &=-\frac{1}{9}-\frac{1}{9}-\frac{1}{18} \\ &=-\frac{5}{18} \end{aligned}$$

A positive determinant indicates a maximum, a negative determinant indicates a minimum.

Problem 6. a. Find the listed partial derivatives of following function.

$$\mathcal{L}(x_1, x_2, \lambda) \;=\; 60x_1 + 24x_2 - \lambda\left(10x_1 + 40x_2 - x_1^2 + x_1x_2 - x_2^2 - 583\right)$$

$\dfrac{\partial\mathcal{L}(x_1,x_2,\lambda)}{\partial x_1} = 60 - 10\lambda + 2\lambda x_1 - \lambda x_2$	$\dfrac{\partial\mathcal{L}(x_1,x_2,\lambda)}{\partial x_2} = 24 - 40\lambda - \lambda x_1 + 2\lambda x_2$	$\dfrac{\partial\mathcal{L}(x_1,x_2,\lambda)}{\partial \lambda} = -10x_1 - 40x_2 + x_1^2 - x_1x_2 + x_2^2 + 583$
$\frac{\partial^2\mathcal{L}}{\partial x_1\partial x_1} = 2\lambda$	$\frac{\partial^2\mathcal{L}}{\partial x_1\partial x_2} = -\lambda$	$-\frac{\partial^2\mathcal{L}}{\partial x_1\partial \lambda} = 10 - 2x_1 + x_2$
$\frac{\partial^2\mathcal{L}}{\partial x_2\partial x_1} = -\lambda$	$\frac{\partial^2\mathcal{L}}{\partial x_2\partial x_2} = 2\lambda$	$-\frac{\partial^2\mathcal{L}}{\partial x_2\partial \lambda} = 40 + x_1 - 2x_2$
$-\frac{\partial^2\mathcal{L}}{\partial \lambda\partial x_1} = 10 - 2x_1 + x_2$	$-\frac{\partial^2\mathcal{L}}{\partial \lambda\partial x_2} = 40 + x_1 - 2x_2$	$\frac{\partial^2\mathcal{L}}{\partial \lambda\partial \lambda} = 0$

b. Show that there are two sets of critical values of the function $\mathcal{L}(x_1, x_2, \lambda)$.
The first set is is $x_1 = 32$, $x_2 = 39$, and $\lambda = -4$.
The second set is $x_1 = 8$, $x_2 = 21$, and $\lambda = 4$.
The objective function is

$$\mathcal{L}(x_1, x_2, \lambda) \;=\; 60x_1 + 24x_2 - \lambda\left(10x_1 + 40x_2 - x_1^2 + x_1x_2 - x_2^2 - 583\right) \tag{12}$$

From equation 12 we can write three equations to find the critical values of $\mathcal{L}$ as follows.

$$60 - 10\lambda + 2\lambda x_1 - \lambda x_2 \;= 0 \tag{13a}$$
$$24 - 40\lambda - \lambda x_1 + 2\lambda x_2 \;= 0 \tag{13b}$$
$$-10x_1 - 40x_2 + x_1^2 - x_1x_2 + x_2^2 + 583 \;= 0 \tag{13c}$$

Rearrange the first equation 13a to obtain

$$\lambda(10 - 2x_1 + x_2) \;=\; 60 \tag{13a$'$}$$

Rearrange the second equation 13b slightly to obtain

$$\lambda(40 + x_1 - 2x_2) \;=\; 24 \tag{13b$'$}$$

Take the ratio of equations 13a′ and 13b′ to obtain

$$\frac{10 - 2x_1 + x_2}{40 + x_1 - 2x_2} \;=\; \frac{60}{24} \;=\; \frac{5}{2} \tag{14}$$

Rearranging and simplifying (14) we obtain

$$\begin{aligned} \frac{10 - 2x_1 + x_2}{40 + x_1 - 2x_2} &= \frac{5}{2} \\ \Rightarrow 200 + 5x_1 - 10x_2 &= 20 - 4x_1 + 2x_2 \\ \Rightarrow 9x_1 &= 12x_2 - 180 \\ \Rightarrow x_1 &= \frac{4}{3}x_2 - 20 \end{aligned} \tag{15}$$

Now substitute the answer for x_1 from equation 15 into equation 13c and multiply out terms as follows. It is easier to work with the negative of 13c so we will do so.

$$\begin{aligned} 10x_1 + 40x_2 - x_1^2 + x_1 x_2 - x_2^2 - 583 &= 0 \\ \Rightarrow 10\left(\frac{4}{3}x_2 - 20\right) + 40x_2 - \left(\frac{4}{3}x_2 - 20\right)^2 + \left(\frac{4}{3}x_2 - 20\right)x_2 - x_2^2 - 583 &= 0 \\ \Rightarrow \frac{40}{3}x_2 - 200 + 40x_2 - \frac{16}{9}x_2^2 + \frac{160}{3}x_2 - 400 + \frac{4}{3}x_2^2 - 20x_2 - x_2^2 - 583 &= 0 \end{aligned} \tag{16}$$

Now multiply equation 16 by 9 and simplify

$$\begin{aligned} \frac{40}{3}x_2 - 200 + 40x_2 - \frac{16}{9}x_2^2 + \frac{160}{3}x_2 - 400 + \frac{4}{3}x_2^2 - 20x_2 - x_2^2 - 583 &= 0 \\ \Rightarrow 120x_2 - 1800 + 360x_2 - 16x_2^2 + 480x_2 - 3600 + 12x_2^2 - 180x_2 - 9x_2^2 - 5247 &= 0 \\ \Rightarrow 780x_2 - 10647 - 13x_2^2 &= 0 \\ \Rightarrow (-13)(819 - 60x2 + x_2^2) &= 0 \end{aligned} \tag{17}$$

Now factor the quadratic in 17 to obtain

$$\begin{aligned} x_2^2 - 60x_2 + 819 &= 0 \\ \Rightarrow (x_2 - 21)(x_2 - 39) &= 0 \\ \Rightarrow x_2 = 21 \text{ and } x_2 &= 39 \end{aligned} \tag{18}$$

Now substitute x_2 into equation 15 to obtain values for x_1.

$$\begin{aligned} x_1 &= \frac{4}{3}x_2 - 20 \\ \Rightarrow x_1 &= \frac{4}{3}(21) - 20 = 28 - 20 = 8 \\ \Rightarrow x_1 &= \frac{4}{3}(39) - 20 = 52 - 20 = 32 \end{aligned} \tag{19}$$

We can obtain λ by substituting x_1 and x_2 into equation 13a′ as follows.

$$
\begin{aligned}
\lambda(10 - 2x_1 + x_2) &= 60 \\
\Rightarrow \lambda &= \frac{60}{10 - 2x_1 + x_2} \\
&= \frac{60}{10 - 2(32) + 39} = \frac{60}{-15} = -4 \\
&= \frac{60}{10 - 2(8) + 21} = \frac{60}{15} = 4
\end{aligned}
\qquad (13a'')
$$

c. Substitute the first set of values for x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is 2,808.

$$\begin{vmatrix} \begin{aligned} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} &= 2\lambda \\ &= 2(-4) \\ &= -8 \end{aligned} & \begin{aligned} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} &= -\lambda \\ &= -(-4) \\ &= 4 \end{aligned} & \begin{aligned} -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} &= 10 - 2x_1 + x_2 \\ &= 10 - (2)(32) + 39 \\ &= -15 \end{aligned} \\ \begin{aligned} \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} &= -\lambda \\ &= -(-4) \\ &= 4 \end{aligned} & \begin{aligned} \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} &= 2\lambda \\ &= 2(-4) \\ &= -8 \end{aligned} & \begin{aligned} -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} &= 40 + x_1 - 2x_2 \\ &= 40 + 32 - (2)(39) \\ &= -6 \end{aligned} \\ \begin{aligned} -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} &= 10 - 2x_1 + x_2 \\ &= 10 - (2)(32) + 39 \\ &= -15 \end{aligned} & \begin{aligned} -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} &= 40 + x_1 - 2x_2 \\ &= 40 + 32 - (2)(39) \\ &= -6 \end{aligned} & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0 \end{vmatrix}$$

We can compute the determinant using the cofactor method and the last row.

The cofactor of (-15) in the 31 place is $(-1)^4((4)(-6) - (-8)(-15)) = -24 - 120 = -144$.
The cofactor of (-6) in the 32 place is $(-1)^5((-8)(-6) - (4)(-15)) = 48 + 60 = -108$.

The determinant is then (-15)(-144) + (-6)(-108)=2160+648=2808.
A positive determinant indicates a maximum, a negative determinant indicates a minimum so (32,39,-4) is a maximum point.

d. Substitute the second set of values for x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is -2,808.

$$\left|\begin{array}{ccc}
\begin{aligned}\frac{\partial^2\mathcal{L}}{\partial x_1\partial x_1} &= 2\lambda \\ &=2(4) \\ &=8\end{aligned} &
\begin{aligned}\frac{\partial^2\mathcal{L}}{\partial x_1\partial x_2} &= -\lambda \\ &= -(4)\end{aligned} &
\begin{aligned}-\frac{\partial^2\mathcal{L}}{\partial x_1\partial \lambda} &=10-2x_1+x_2 \\ &=10-(2)(8)+21 \\ &=15\end{aligned} \\
\begin{aligned}\frac{\partial^2\mathcal{L}}{\partial x_2\partial x_1} &= -\lambda \\ &= -(4)\end{aligned} &
\begin{aligned}\frac{\partial^2\mathcal{L}}{\partial x_2\partial x_2} &= 2\lambda \\ &=2(4) \\ &=8\end{aligned} &
\begin{aligned}-\frac{\partial^2\mathcal{L}}{\partial x_2\partial \lambda} &=40+x_1-2x_2 \\ &=40+8-(2)(21) \\ &=6\end{aligned} \\
\begin{aligned}-\frac{\partial^2\mathcal{L}}{\partial \lambda\partial x_1} &=10-2x_1+x_2 \\ &=10-(2)(8)+21 \\ &=15\end{aligned} &
\begin{aligned}-\frac{\partial^2\mathcal{L}}{\partial \lambda\partial x_2} &=40+x_1-2x_2 \\ &=40+8-(2)(21) \\ &=6\end{aligned} &
\frac{\partial^2\mathcal{L}}{\partial \lambda\partial \lambda} = 0
\end{array}\right|$$

We can compute the determinant using the cofactor method and the last row.

The cofactor of (15) in the 31 place is $(-1)^4((-4)(6)-(8)(15)) = -24-120 = -144.$
The cofactor of (6) in the 32 place is $(-1)^5((8)(6)-(-4)(15)) = -(48+60) = -108.$

The determinant is then (15)(-144) + (6)(-108)=-2160+-648=-2808.

A negative determinant indicates a minimum so (8,21,4) is a minimum.