ECONOMICS 207
SPRING 2007
LABORATORY EXERCISE 15

For your information, the Hessian matrix in the profit maximization problem written as

$$\pi(x_1,\, x_2) \;=\; pf(x_1,\, x_2) \;-w_1x_1 - w_2x_2$$

is given by

$$H(\pi(x_1,\, x_2)) = \begin{bmatrix} \dfrac{\partial^2 \pi(x_1, x_2)}{\partial x_1 \partial x_1} & \dfrac{\partial^2 \pi(x_1, x_2)}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 \pi(x_1, x_2)}{\partial x_2 \partial x_1} & \dfrac{\partial^2 \pi(x_1, x_2)}{\partial x_2 \partial x_2} \end{bmatrix}$$

The bordered Hessian in the constrained optimization problem written as

$$\mathfrak{L}(x_1, x_2, \lambda) \;=\; f(x_1, x_2) - \lambda g(x_1, x_2)$$

is given by

$$H_B = \begin{bmatrix} \dfrac{\partial^2 \mathfrak{L}(x_1, x_2, \lambda)}{\partial x_1 \partial x_1} & \dfrac{\partial^2 \mathfrak{L}(x_1, x_2, \lambda)}{\partial x_1 \partial x_2} & \dfrac{\partial g(x_1, x_2)}{\partial x_1} \\ \dfrac{\partial^2 \mathfrak{L}(x_1, x_2, \lambda)}{\partial x_2 \partial x_1} & \dfrac{\partial^2 \mathfrak{L}(x_1, x_2, \lambda)}{\partial x_2 \partial x_2} & \dfrac{\partial g(x_1, x_2)}{\partial x_2} \\ \dfrac{\partial g(x_1, x_2)}{\partial x_1} & \dfrac{\partial g(x_1, x_2)}{\partial x_2} & 0 \end{bmatrix}$$

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Problem 1. The cost function for a firm is a rule or mapping that tells the total cost of production of any output level produced by the firm. If y represents the output of the firm, then the cost function is given by c(y). Consider the following competitive firm.

$$price \;=\; p \;= 419$$

$$cost \;=\; c(y) \;= 300 \;+\; 500y \;-\; 45y^2 \;+\; 3y^3$$

a. Write a function representing the profit of the firm as a function of the given output price and output level.

$$\begin{aligned}\pi =& py - c(y)\\ =& 419y - 300 - 500y + 45y^2 - 3y^3\\ =& -300 - 81y + 45y^2 - 2y^3\end{aligned}$$

b. What is the profit maximizing level of output for this firm? Verify that the output level you choose is the profit maximizing point.

We first find the first and second derivative.

$$\begin{aligned}\pi' =& 0\\ \Rightarrow -81 + 90y - 9y^2 =& 0\\ \pi'' =& 90 - 18y\end{aligned}$$

Then set the first derivative to 0.

$$\begin{aligned}\pi' =& 0\\ \Rightarrow -81 + 90y - 9y^2 =& 0\\ \Rightarrow -9 + 10y - y^2 =& 0\\ \Rightarrow y^2 - 10y + 9 =& 0\\ \Rightarrow (y-1)(y-9) =& 0\\ \Rightarrow y =& 1, 9.\end{aligned}$$

Next, we check the second derivative to find the optimal level of output.

$$\begin{aligned}\pi''(1) =& 90 - 18\\ =& 72 > 0\\ \pi''(9) =& 90 - 162\\ =& -72\end{aligned}$$

Hence, 9 is the profit maximizing level of output.

Problem 2. Consider a competitive firm with the following production function where y represents the level of output of the firm and x represents the level of the single variable input.

$$y \;=\; f(x) \;=\; 20x \;+\; 60x^2 \;-\; 2x^3$$

The firm faces an output price of p = 10 and an input price of w = 4040.

a. Write a function representing the profit of the firm as a function of the output and input prices and the input level.

$$\begin{aligned}\pi =& py - wx \\ =& 10(20x + 60x^2 - 2x^3) - 4040x \\ =& 200x + 600x^2 - 20x^3 - 4040x \\ =& -3840x + 600x^2 - 20x^3\end{aligned}$$

b. What is the profit maximizing level of input for this firm? Verify that the input level you choose is the profit maximizing point.

First, we find the first and second derivatives.

$$\begin{aligned}\pi' =& -3840 + 120x - 60x^2 \\ \pi'' =& 120 - 120x\end{aligned}$$

Then, we set the first derivative to 0 to find the optimal level of inputs.

$$\begin{aligned}\pi' =& 0 \\ \Rightarrow -3840 + 120x - 60x^2 =& 0 \\ \Rightarrow 64 - 20x + x^2 =& 0 \\ \Rightarrow (x - 16)(x - 4) =& 0 \\ \Rightarrow x =& 16, 4.\end{aligned}$$

Finally, we plug in the above levels to the second derivative to check which one is the optimal one.

$$\begin{aligned}\pi''(4) =& 1200 - 120 \times 4 \\ =& 720 > 0 \\ \pi''(16) =& 1200 - 120 \times 16 \\ =& -720\end{aligned}$$

Since $\pi''(16) < 0$, 16 is the profit maximizing level of input.

c. What is the marginal product (MP_x) of the variable input at the profit maximizing input level?

$$\begin{aligned} MP_x =& f'(x) \\ =& 20 + 120x - 6x^2 \\ MP_x|_{16} =& 20 + 120(16) - 6(16)^2 \\ =& 20 + 1920 - 1536 \\ =& 404 \end{aligned}$$

d. Verify that pMP_x = w at the profit maximizing input level.

$$pMP_x = 10(404) = 4040 = w$$

Problem 3. Below you are given a production function for a competitive firm. You are also given the price of the firm's output and the prices of the two inputs used by the firm. Output price is represented by p, the price of the first input by w_1 and the price of the second input by w_2.

$$f(x_1, x_2) = 50x_1 + 30x_2 - x_1^2 + x_1x_2 - 2x_2^2$$
$$p = 10$$
$$w_1 = 300, \qquad w_2 = 50$$

a. Write an equation that represents profit as a function of the two inputs x_1 and x_2. Simplify the expression.

$$\begin{aligned}\pi &= pf - wx_1 - wx_2\\ &= 10(50x_1 + 30x_2 - x_1^2 + x_1x_2 - 2x_2^2) - 300x_1 - 50x_2\\ &= 500x_1 + 300x_2 - 10x_1^2 + 10x_2x_2 - 20x_2^2 - 300x_1 - 50x_2\\ &= 200x_1 + 250x_2 - 10x_1^2 + 10x_1x_2 - 20x_2^2\end{aligned}$$

b. Find all first and second partial derivatives of the function.

$\frac{\partial \pi}{\partial x_1} = 200 - 20x_1 + 10x_2$	$\frac{\partial \pi}{\partial x_2} = 250 + 10x_1 - 40x_2$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -20$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 10$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 10$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -40$

c. Find potential profit maximizing levels of x_1 and x_2.

We set the first derivatives to 0.

$$\frac{\partial \pi}{\partial x_1} = 0$$
$$\frac{\partial \pi}{\partial x_2} = 0$$

$\Rightarrow$

$$200 - 20x_1 + 10x_2 = 0,$$
$$250 + 10x_1 - 40x_2 = 0,$$

which can be simplified as

$$20 - 2x_1 + x_2 = 0,$$
$$25 + x_1 - 4x_2 = 0.$$

To solve the above equations, we multiply the second one by 2 and add it to the first one.

$$70 - 7x_2 = 0$$
$$\Rightarrow x_2 = 10$$

Plugging that into the second equation , $x_1 = 4x_2 - 25$

$$= 15.$$

d. Fill in the elements of the Hessian matrix of the profit equation and then verify the optimal levels of x_1 and x_2.

$$\begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -20 & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 10 \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 10 & \frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -40 \end{vmatrix} = 800 - 100 = 700 > 0$$

Since $\frac{\partial^2 \pi}{\partial x_1 \partial x_1} > 0$ and the determinant is positive, $(15, 10)$ is the profit maximizing level.

Problem 4. Below you are given a production function for a competitive firm. You are also given the price of the firm's output and the prices of the two inputs used by the firm. Output price is represented by p, the price of the first input by w_1 and the price of the second input by w_2.

$$f(x_1, x_2) = x_1^{1/3} x_2^{1/4}$$
$$p = 864$$
$$w_1 = 64, \qquad w_2 = 81$$

a. Write an equation that represents profit as a function of the two inputs x_1 and x_2. Simplify the expression.

$$\pi = pf - w_1 x_1 - w_2 x_2$$
$$= 864 x_1^{1/3} x_2^{1/4} - 64 x_1 - 81 x_2$$

b. Find all first and second partial derivatives of the function.

$\frac{\partial \pi}{\partial x_1} = 288 x_1^{-2/3} x_2^{1/4} - 64$	$\frac{\partial \pi}{\partial x_2} = 216 x_1^{1/3} x_2^{-3/4} - 81$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -192 x_1^{-5/3} x_2^{1/4}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 72 x_1^{-2/3} x_2^{-3/4}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 72 x_1^{-2/3} x_2^{-3/4}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -162 x_1^{1/3} x_2^{-7/4}$

c. Find potential profit maximizing levels of x_1 and x_2.

$$\frac{\partial \pi}{\partial x_1} = 0$$
$$\frac{\partial \pi}{\partial x_2} = 0$$

$\Rightarrow$

$$288 x_1^{-2/3} x_2^{1/4} - 64 = 0$$
$$216 x_1^{1/3} x_2^{-3/4} - 81 = 0.$$

To solve the above equations, we move the second term to the right-hand sides and divide two equations.

$$\frac{288 x_2}{216 x_1} = \frac{64}{81}$$
$$\Rightarrow \frac{x_2}{x_1} = \frac{64 \cdot 216}{81 \cdot 288}$$
$$\Rightarrow \frac{x_2}{x_1} = \frac{16}{27}$$
$$\Rightarrow x_2 = \frac{16}{27} x_1$$

We next substitute the above equation into the second one.

$$216 x_1^{1/3} (\frac{16}{27} x_1)^{-3/4} - 81 = 0$$
$$\Rightarrow 216 x_1^{1/3} (\frac{16}{27} x_1)^{-3/4} = 81$$
$$\Rightarrow 8 x_1^{1/3-3/4} (2^4 3^{-3})^{-3/4} = 3$$
$$\Rightarrow 82^{-}3 x_1^{-5/12} 3^{9/4} = 3$$
$$\Rightarrow 3^{9/4-1} = x_1^{5/4}$$
$$\Rightarrow 3^{5/4} = x_1^{5/4}$$
$$\Rightarrow x_1 = 3^3 = 27$$
$$\text{and } x_2 = \frac{16}{27} x_1 = 16.$$

d. Fill in the elements of the Hessian matrix of the profit equation evaluated at the critical values of x_1 and x_2 and then verify the optimal levels of x_1 and x_2.

$$\begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -192(3)^{-5}2 & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 72(3)^{-2}(2)^{-3} \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 72(3)^{-2}(2)^{-3} & \frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -162(3)(2)^{-7} \end{vmatrix} = 5 > 0$$

Since $\frac{\partial^2 \pi}{\partial x_1 \partial x_1} < 0$ and determinant is positive, $(27, 16)$ is the profit maximizing level.

e. What is the optimal level of output?

$$\begin{aligned} f =& x_1^{1/3} x_2^{1/4} \\ =& 27^{1/3} 16^{1/4} \\ =& 3(2) = 6 \end{aligned}$$

Problem 5. Consider a firm with a production function given by

$$f(x_1, x_2) \;=\; x_1^{1/3} x_2^{1/4}$$

The firm faces prices and an output constraint given by

$$\begin{aligned} w_1 &= 64 \\ w_2 &= 81 \\ y_0 &= 6 \end{aligned}$$

Find potential levels of x_1, x_2 and λ to minimize cost for this firm given the output constraint and the stated prices. Verify that these input levels minimize cost.

a. Set up the objective function for this problem and find all first and second partial derivatives of the function with respect to x_1 and x_2.

$$\mathcal{L}(x_1, x_2, \lambda) \;=\; 64x_1 + 81x_2 - \lambda(x_1^{1/3} x_2^{1/4} - 6)$$

$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 64 - \lambda \frac{1}{3} x_1^{-2/3} x_2^{1/4}$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = 81 - \lambda \frac{1}{4} x_1^{1/3} x_2^{-3/4}$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = \frac{2}{9} \lambda x_1^{-5/3} x_2^{1/4}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = -\frac{1}{12} \lambda x_1^{-2/3} x_2^{-3/4}$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = -\frac{1}{12} \lambda x_1^{-2/3} x_2^{-3/4}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = \frac{3}{16} \lambda x_1^{1/3} x_2^{-7/4}$

b. What is the derivative of the objective function in this problem with respect to λ?

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -x_1^{1/3} x_2 1/4 + 6$$

c. Find the partial derivatives of the constraint equation with respect to x_1 and x_2.

$\frac{\partial g(x_1, x_2)}{\partial x_1} = \frac{1}{3} x_1^{-2/3} x_2^{1/4}$	$\frac{\partial g(x_1, x_2)}{\partial x_2} = \frac{1}{4} x_1^{1/3} x_2^{-3/4}$

d. Use the information from 5a and 5b to find critical values for x_1, x_2 and λ.

$$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 0$$
$$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = 0$$

$\Rightarrow$

$$\frac{64}{81} = \frac{4x_2}{3x_1}$$
$$\Rightarrow \frac{x_2}{x_1} = \frac{16}{27}$$

Plugging the above equation to the constraint, we have

$$x_1^{1/3} x_2 1/4 = 6$$
$$\Rightarrow x_1^{1/3} (\frac{16}{27} x_1)^{1/4} = 6$$
$$\Rightarrow x_1^{1/3+1/4} 2(3)^{-3/4} = 6$$
$$\Rightarrow x_1^{7/12} = 3^{1+3/4} = 3^{7/4}$$
$$\Rightarrow x_1 = 3^3 = 27$$
$$\text{and } x_2 = 16$$

And using $\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 0$, we have

$$64 - \lambda \frac{1}{3} x_1^{-2/3} x_2^{1/4} = 0$$
$$\Rightarrow 64 - \frac{1}{3} 3^{-2} 2\lambda = 0$$
$$\Rightarrow \lambda = 32 \cdot 3^3 = 864$$

e. Use the answers from part 5d and the expressions from parts 5a and 5c to fill in the bordered Hessian matrix for this problem. Then determine whether the critical values indicate a maximum or a minimum.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = \frac{128}{81} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = -1 & \frac{\partial g(x_1, x_2)}{\partial x_1} = \frac{2}{27} \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = -1 & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = \frac{243}{64} & \frac{\partial g(x_1, x_2)}{\partial x_2} = \frac{3}{32} \\ \frac{\partial g(x_1, x_2)}{\partial x_1} = \frac{2}{27} & \frac{\partial g(x_1, x_2)}{\partial x_2} = \frac{3}{32} & 0 \end{vmatrix} = -\frac{1}{18} < 0$$

Hence, $x_1 = 27, x_2 = 16$ are the cost minimizing level of inputs.

f. What is the cost for this firm to produce 6 units of output?

$$Cost = 64 \cdot 27 + 81 \cdot 16 = 3024$$

g. Relate the optimal value of λ from part 5d to the price from problem 4.

The λ here is equal to the price from problem 4.

Problem 6. Consider a firm with a production function given by

$$f(x_1, x_2) \;=\; x_1^{1/3} x_2^{1/4}$$

The firm faces prices and a cost constraint given by

$$\begin{aligned} w_1 &= 64 \\ w_2 &= 81 \\ c_0 &= 3024 \end{aligned}$$

Find potential levels of x_1, x_2 and λ to maximize output for this firm given the cost constraint and the stated prices. Verify that these input levels maximize output.

a. Set up the objective function for this problem and find all first and second partial derivatives of the function with respect to x_1 and x_2.

$$\mathcal{L}(x_1, x_2, \lambda) \ = \ x_1^{1/3} x_2^{1/4} - \lambda(64x_1 + 81x_2 - 3024)$$

$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = \frac{1}{3} x_1^{-2/3} x_2^{1/4} - 64\lambda$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = \frac{1}{4} x_1^{1/3} x_2^{-3/4} - 81\lambda$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -\frac{2}{9} x_1^{-5/3} x_2^{1/4}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = \frac{1}{12} x_1^{-2/3} x_2^{-3/4}$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = \frac{1}{12} x_1^{-2/3} x_2^{-3/4}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -\frac{3}{16} x_1^{1/3} x_2^{-7/4}$

b. What is the derivative of the objective function in this problem with respect to λ?

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(64x_1 + 81x_2 - 3024)$$

c. Find the partial derivatives of the constraint equation with respect to x_1 and x_2.

$\frac{\partial g(x_1, x_2)}{\partial x_1} = 64$	$\frac{\partial g(x_1, x_2)}{\partial x_2} = 81$

d. Use the information from 6a and 6b to find critical values for x_1, x_2 and λ.

We set the first derivative of $\mathcal{L}$ to 0.

$$\frac{1}{3}x_1^{-2/3}x_2^{1/4} = 64\lambda$$

$$\frac{1}{4}x_1^{1/3}x_2^{-3/4} = 81\lambda$$

$$\Rightarrow \frac{4x_2}{3x_1} = \frac{64}{81}$$

$$\Rightarrow x_2 = \frac{16}{27}x_1$$

Plugging that into part b) $\Rightarrow 64x_1 + 48x_1 = 3024$

$$\Rightarrow x_1 = 27$$

and $x_2 = 16$

Plugging x_1 and x_2 into part a) $\Rightarrow \frac{1}{3}\frac{1}{9}2 = 64\lambda$

$$\Rightarrow \lambda = \frac{1}{864}$$

Hence, $x_1 = 27, x_2 = 16$ are the output maximizing level.

e. Use the answers from part 6d and the expressions from parts 6a and 6c to fill in the bordered Hessian matrix for this problem. Then determine whether the critical values indicate a maximum or a minimum. The determinant of the bordered Hessian is 42.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -\frac{4}{2187} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = \frac{1}{864} & \frac{\partial g(x_1, x_2)}{\partial x_1} = 64 \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = \frac{1}{864} & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -\frac{9}{2048} & \frac{\partial g(x_1, x_2)}{\partial x_2} = 81 \\ \frac{\partial g(x_1, x_2)}{\partial x_1} = 64 & \frac{\partial g(x_1, x_2)}{\partial x_2} = 81 & 0 \end{vmatrix} = 42 > 0$$

f. How much output can this firm produce given it spends only $3024?

$$f = 3 \cdot 2 = 6$$

g. Why did you not need to do any of the work in problem 6 given you had already worked problems 4 and 5.

That is because these problems are equivalent.

Problem 7. Consider a firm with a production function given by

$$f(x_1, x_2) \;=\; 100x_1 + 300x_2 - 2x_1^2 + 2x_1x_2 - x_2^2$$

The firm faces prices and a cost constraint given by

$$w_1 \;=\; 150, \qquad w_2 \;=\; 108, \qquad c_0 \;=\; 54762$$

Find potential levels of x_1, x_2 and λ to maximize output for this firm given the cost constraint and the stated prices. Verify that these input levels maximize output.

a. Set up the objective function for this problem and find all first and second partial derivatives of the function with respect to x_1 and x_2.

b. $\mathcal{L}(x_1, x_2, \lambda) = 100x_1 + 300x_2 - 2x_1^2 + 2x_1x_2 - x_2^2 - \lambda(150x_1 + 108x_2 - 54762)$

$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 100 - 4x_1 + 2x_2 - 150\lambda$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = 300 + 2x_1 - 2x_2 - 108\lambda$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -4$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = 2$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = 2$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -2$

c. What is the derivative of the objective function in this problem with respect to λ?

$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(150x_1 + 108x_2 - 54762)$$

d. Find the partial derivatives of the constraint equation with respect to x_1 and x_2.

$\frac{\partial g(x_1,x_2)}{\partial x_1} = 150$	$\frac{\partial g(x_1,x_2)}{\partial x_2} = 108$

e. Use the information from 7a and 7c to find critical values for x_1, x_2 and λ.

First, we set the first derivatives to 0.

$$\begin{aligned} 100 - 4x_1 + 2x_2 &= 150\lambda \\ 300 + 2x_1 - 2x_2 &= 108\lambda \end{aligned}$$

that is equivalent to

$$\begin{aligned} \frac{100 - 4x_1 + 2x_2}{300 + 2x_1 - 2x_2} &= \frac{150}{108} \\ \Rightarrow \frac{50 - 2x_1 + x_2}{150 + x_1 - x_2} &= \frac{150}{108} \\ \Rightarrow 150^2 + 150x_1 - 150x_2 &= 50 \cdot 108 - 216x_1 + 108x_2 \\ \Rightarrow 366x_1 - 258x_2 &= -17100 \\ \Rightarrow x_1 &= \frac{43}{61}x_2 - \frac{2850}{61} \end{aligned} \tag{1}$$

We substitute (1) into the constraint, then we have

$$\begin{aligned} 150x_1 + 108x_2 &= 54762 \\ \Rightarrow 150(\frac{43}{61}x_2 - \frac{2850}{61}) + 108x_2 &= 54762 \\ \Rightarrow \frac{13038}{61}x_2 &= \frac{3767982}{61} \\ \Rightarrow x_2 &= 289 \\ \text{and } x_1 &= 157. \end{aligned}$$

Finally, we plug x_1, x_2 into the equation $\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = 0$, that is,

$$\begin{aligned} 108\lambda =& 300 + 2(x_1 - x_2) \\ =& 300 - 264 = 36 \\ \Rightarrow \lambda =& \frac{1}{3} \end{aligned}$$

f. Use the answers from part 7e and the expressions from parts 7a and 7d to fill in the bordered Hessian matrix for this problem. Then determine whether the critical values indicate a maximum or a minimum.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -4 & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = 2 & \frac{\partial g(x_1, x_2)}{\partial x_1} = 150 \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = 2 & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -2 & \frac{\partial g(x_1, x_2)}{\partial x_2} = 108 \\ \frac{\partial g(x_1, x_2)}{\partial x_1} = 150 & \frac{\partial g(x_1, x_2)}{\partial x_2} = 108 & 0 \end{vmatrix} = 156456 > 0$$

g. How much output can this firm produce given it spends only \$54762?

$$\begin{aligned} f &= 100x_1 + 300x_2 - 2x_1^2 + 2x_1x_2 - x_2^2 \\ &= 100 \times 157 + 300 \times 289 - 2(157)^2 + 2 \times 157 \times 289 - 289^2 \\ &= 60327 \end{aligned}$$