ECONOMICS 207
SPRING 2007
PROBLEM SET 10—KEY

Problem 1. For each of the following systems of equations, find the solution vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ or $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ by appending the right-hand side vector to the coefficient matrix and performing row reduction.

a.

$$\begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

We expand the right-hand side to the coefficient matrnix, and we have

$$\begin{bmatrix} -1 & 1 & -2 \\ 2 & -3 & 2 \end{bmatrix}.$$

First, multiply the first row by -1, that is

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 2 \end{bmatrix}.$$

Next, we multiply the first row by 2 and subtract it from the second row, which results in

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & -2 \end{bmatrix}.$$

Multiply the second row by -1 and add it to the first row, which gives

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \end{bmatrix}.$$

So, the solution is

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

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b.

$$\begin{pmatrix} 2 & -1 & 4 \\ 1 & 0 & 2 \\ 4 & -1 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 11 \\ 5 \\ 19 \end{pmatrix}$$

We expand the right-hand side to the coefficient matrnix, and we have

$$\begin{pmatrix} 2 & -1 & 4 & 11 \\ 1 & 0 & 2 & 5 \\ 4 & -1 & 7 & 19 \end{pmatrix}.$$

First, subtract the last row by the first row multiplied by 2, we get

$$\begin{pmatrix} 2 & -1 & 4 & 11 \\ 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 \end{pmatrix}.$$

Then multiply the first row again by $1/2$ and subtract that from the second row, that is,

$$\begin{pmatrix} 2 & -1 & 4 & 11 \\ 0 & 1/2 & 0 & -1/2 \\ 0 & 1 & -1 & -3 \end{pmatrix}.$$

Multiply the second row by 1 is

$$\begin{pmatrix} 2 & -1 & 4 & 11 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & -3 \end{pmatrix}.$$

Then, subtract the last row by the second one, we get,

$$\begin{pmatrix} 2 & -1 & 4 & 11 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -2 \end{pmatrix}.$$

Then, adding the second row to the first one, and multiply the last row by -1,

$$\begin{pmatrix} 2 & 0 & 4 & 10 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

Next, multiply the last row by 4 and subtract that from the first row, which results in

$$\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

At last, we divide the first row by 2,

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

Hence, the solution is

$$\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

Problem 2. Consider the following matrices.

$$A = \begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ -3 & -4 & -2 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & -3 & 2 \\ -2 & 5 & -2 \\ 4 & -11 & 7 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 7 \end{bmatrix}$$

a.

i Find the determinant of A.

$$\text{det(A)} = (-1)(-3) - 2 = 1$$

ii Find the inverse of A using the adjoint method.

$$\text{Cofactor of A} = \begin{pmatrix} -3 & -2 \\ -1 & -1 \end{pmatrix}$$

$$\Rightarrow \text{Adjoint matrix} = \begin{pmatrix} -3 & -1 \\ -2 & -1 \end{pmatrix}$$

Since the determinant of A is 1, the inverse of A is the same as the above.

iii Find the inverse of A using row reduction.

We start to work on the following matrix:

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ -2 & -3 & 0 & 1 \end{pmatrix}.$$

First, add the second row by the first one multiplied by 2, which gives us

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{pmatrix}.$$

Next, add the second row to the first one,

$$\begin{pmatrix} -1 & 0 & 3 & 1 \\ 0 & -1 & 2 & 1 \end{pmatrix}.$$

Last, multiply the first and second both by -1, which results in

$$\begin{pmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & -2 & -1 \end{pmatrix}.$$

Hence, the inverse of A is

$$\begin{pmatrix} -3 & -1 \\ -2 & -1 \end{pmatrix}.$$

b.

i Find the determinant of C.

$$\det(C) = 1 - 0 = 1$$

ii Find the inverse of C using the adjoint method.

$$\text{Cofactor(C)} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix},$$
$$\text{Adjoint(C)} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

And A^{-1} is then

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix},$$

since the determinant is 1.

iii Find the inverse of C using row reduction.

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix},$$

Multiply the first by 2 and subtract it from the second row, we get

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix}.$$

Then the inverse of C is

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

c.

i Find the determinant of D.

$$\text{det(D)} = -10 - 12 - 8 + 15 + 8 + 8 = 1$$

ii Find the inverse of D using the adjoint method.

$$\text{Cofactor(D)} = \begin{pmatrix} (-10+8) & -(-4+6) & (-8+15) \\ -(-4+4) & -2+3 & -(-4+6) \\ 4-5 & -(2-2) & 5-4 \end{pmatrix},$$

$$= \begin{pmatrix} -2 & -2 & 7 \\ 0 & 1 & -2 \\ -1 & 0 & 1 \end{pmatrix},$$

$$\text{Adjoint(D)} = \begin{pmatrix} -2 & 0 & -1 \\ -2 & 1 & 0 \\ 7 & -2 & 1 \end{pmatrix}.$$

$$D^{-1} = \frac{Adjoint(D)}{det(D)}$$

$$= \begin{pmatrix} -2 & 0 & -1 \\ -2 & 1 & 0 \\ 7 & -2 & 1 \end{pmatrix}.$$

iii Find the inverse of D using row reduction.

We start with following matrix by expending D:

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 5 & 2 & 0 & 1 & 0 \\ -3 & -4 & -2 & 0 & 0 & 1 \end{pmatrix}.$$

First, subtracting the second row by the first one multiplied by 2,

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ -3 & -4 & -2 & 0 & 0 & 1 \end{pmatrix};$$

Next, multiply the first row by 3 and add it to the third row, and we have

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 2 & 1 & 3 & 0 & 1 \end{pmatrix}.$$

Next, we multiply the second row by 2 and subtract that from the third row,

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & 1 \end{pmatrix}.$$

Then, multiplying the second row by 2 and subtract that from 1, we get,

$$\begin{pmatrix} 1 & 0 & 1 & 5 & -2 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & 1 \end{pmatrix}.$$

And finally, subtracting the first row by the last row, which results in

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 & -1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & 1 \end{pmatrix}.$$

Hence, the inverse of D is

$$\begin{pmatrix} -2 & 0 & -1 \\ -2 & 1 & 0 \\ 7 & -2 & 1 \end{pmatrix}.$$

d.

i Find the determinant of F.

$$\begin{aligned} det(F) =& 1 * 3 * 7 + 2 * 5 * 3 + 3 * 2 * 5 - 3 * 3 * 3 - 2 * 2 * 7 - 1 * 5 * 5 \\ =& 1 \end{aligned}$$

ii Find the inverse of F using the adjoint method.

$$\text{Cofactor(F)} = \begin{pmatrix} -4 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$
$$\text{Adjoint(F)} = \begin{pmatrix} -4 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

And F^{-1} is the same as the above.

iii Find the inverse of F using row reduction.

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 5 & 0 & 1 & 0 \\ 3 & 5 & 7 & 0 & 0 & 1 \end{pmatrix}$$

First, $(2) - (1) * 2 \rightarrow$

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 3 & 5 & 7 & 0 & 0 & 1 \end{pmatrix},$$

next, $(3) - (1) * 3 \rightarrow$

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & -1 & -2 & -3 & 0 & 1 \end{pmatrix}.$$

Then, we subtract the second row from the third row, giving

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{pmatrix},$$

and subtract the third row from the second one,

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{pmatrix}.$$

We next multiply the second row by 2 and add it to the first one,

$$\begin{pmatrix} 1 & 0 & 3 & -1 & 4 & -2 \\ 0 & -1 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{pmatrix},$$

and multiply the third row by 3 and add it to the first row,

$$\begin{pmatrix} 1 & 0 & 0 & -4 & 1 & -1 \\ 0 & -1 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{pmatrix}.$$

Finally, multiply the second and the third row both by -1, which gives us

$$\begin{pmatrix} 1 & 0 & 0 & -4 & 1 & -1 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{pmatrix}.$$

So, F^{-1} is

$$\begin{pmatrix} -4 & 1 & -1 \\ 1 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Problem 3. For each of the following problems, find the critical points. For each critical point state whether the function is at a relative maximum, relative minimum, or otherwise. Check to see if there are points of inflection **at points other than** critical points.

a. $f(x) = \frac{3x^4}{4} - x^3 - 9x^2$

$$\begin{aligned} f'(x) =& 3x^3 - 3x^2 - 18x \\ f''(x) =& 9x^2 - 6x - 18 \end{aligned}$$

First, we solve the critical points:

$$\begin{aligned} f'(x) =& 0, \\ \rightarrow 3x^3 - 3x^2 - 18x =& 0, \\ \rightarrow x^3 - x^2 - 6x =& 0, \\ \rightarrow x(x-3)(x+2) =& 0, \\ \rightarrow x = 0, 3, -2. \end{aligned}$$

Next we check at which point the function is a maximum or minimum.

$$\begin{aligned} f''(0) =& -18 < 0, \\ f''(3) =& 45 > 0, \\ f''(-2) =& 30 > 0. \end{aligned}$$

Hence, 0 is a maximum point, and 3 and -2 are minimum points.

Finally, we check if there are inflection points other than critical points.

$$\begin{aligned} f''(x) =& 0, \\ \rightarrow 3x^2 - 2x - 6 =& 0, \\ \rightarrow x = \frac{1 + \pm\sqrt{19}}{3}. \end{aligned}$$

The inflection points are $\frac{1+\pm\sqrt{19}}{3}$.

b. $f(x) = x^2e^{-2x}$

$$f'(x) = 2xe^{-2x} - 2x^2e^{-2x}$$
$$f''(x) = 2e^{-2x} - 8xe^{-2x} + 4x^2e^{-2x}$$

First, we solve the critical points:

$$f'(x) = 0$$
$$2xe^{-2x} - 2x^2e^{-2x} = 0$$
$$\to 2xe^{-2x}(1-x) = 0$$
$$\to x = 0, 1$$

Next, check which one is the maximum or minimum point.

$$f''(0) = 2e^{-2x}[1 - 4x + 2x^2]$$
$$= 2 > 0$$
$$f''(1) = -2e^{-2} < 0$$

Hence, 0 is a minimum point and 1 is the maximum point.

Finally, check if there are any inflection points.

$$f''(x) = 0$$
$$\to 2e^{-2x} - 8xe^{-2x} + 4x^2e^{-2x} = 0,$$
$$\to 2e^{-2x}[1 - 4x + 2x^2] = 0,$$
$$\to 1 - 4x + 2x^2 = 0,$$
$$\to x = \frac{2 \pm \sqrt{2}}{2}.$$

So, the inflection points are $\frac{2\pm\sqrt{2}}{2}$.

c. $f(x) = \frac{3}{4}x^4 - 5x^3 - 3x^2 + 72x$,

$$f'(x) = 3x^3 - 15x^2 - 6x + 72,$$
$$f''(x) = 9x^2 - 30x - 6.$$

First, we solve the critical points:

$$f'(x) = 0,$$
$$3x^3 - 15x^2 - 6x + 72 = 0,$$
$$x^3 - 5x^2 - 2x + 24 = 0,$$
$$(x-3)(x^2 - 2x - 8) = 0,$$
$$(x-3)(x-4)(x+2) = 0,$$
$$x = 3, 4, -2.$$

Next, check the second derivative:

$$f''(3) = -15 < 0,$$
$$f''(4) = 18 > 0,$$
$$f''(-2) = 90 > 0.$$

Hence, 3 is the maximum point, and 4,-2 are minimum points.

Finally, check if there are inflection points:

$$f''(x) = 0,$$
$$\rightarrow 9x^2 - 30x - 6 = 0,$$
$$\rightarrow 3x^2 - 10x - 2 = 0,$$
$$\rightarrow x = \frac{5 \pm \sqrt{31}}{3}.$$

The inflection points are $\frac{5 \pm \sqrt{31}}{3}$.