ECONOMICS 207
SPRING 2007
PROBLEM SET 13

Problem 1. For the following problem, write an equation that represents profit as a function of the input x. Write it in the form π = pf(x)-wx and then simplify the expression. Then find the first and second derivatives of the function. Then find the critical points. For each critical point state whether profit is at a relative maximum, relative minimum, or otherwise. Check to see if there are points of inflection **at points other than** critical points.

$$\begin{aligned} f(x) &= 60x \ + \ 50x^2 \ - \ 2x^3 \\ p &= 2 \\ w &= 692 \end{aligned}$$

Solution:

$$\begin{aligned} \pi =& 2 \cdot (60x \ + \ 50x^2 \ - \ 2x^3) \\ =& 120x + 100x^2 - 4x^3 \\ \frac{\partial \pi}{\partial x} =& 120 + 200x - 12x^2 \\ \frac{\partial^2 \pi}{\partial x^2} =& 200 - 24x \end{aligned}$$

Setting $\frac{\partial \pi}{\partial x}$ to 0, we first solve the critical points:

$$120 + 200x - 12x^2 = 0$$

Divding both sides by -4, we get

$$3x^2 - 50x - 30 = 0$$

Using quadratic formula, we get the following critical points:

$$\begin{aligned} x_1 =& \frac{-25 + \sqrt{715}}{3} \\ x_2 =& \frac{-25 - \sqrt{715}}{3} \end{aligned}$$

Date: 18 April 2008.

We then evaluate $\frac{\partial^2 \pi}{\partial x^2}$ at the critical points, that is,

$$\begin{aligned}\frac{\partial^2 \pi}{\partial x^2}|_{x_1} =& 200 - 24 \cdot \frac{-25 + \sqrt{715}}{3} \\ =& 200 + 8 \cdot 25 - 8\sqrt{715} \\ =& 400 - 8\sqrt{715} < 0 \\ \frac{\partial^2 \pi}{\partial x^2}|_{x_2} =& 200 - 24 \cdot \frac{-25 - \sqrt{715}}{3} \\ =& 200 + 8 \cdot 25 + 8\sqrt{715} \\ =& 400 + 8\sqrt{715} > 0\end{aligned}$$

Hence, x_1 is the maximum point and x_2 is the minimum point.

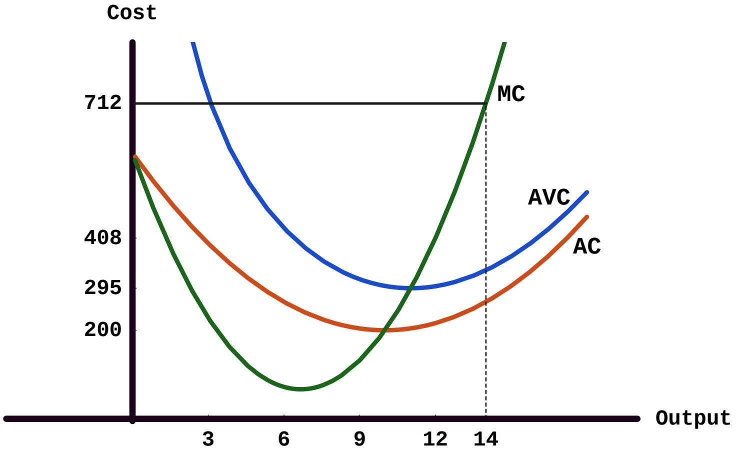
For inflection point, letting $\frac{\partial^2}{\partial x^2} = 0$, we get

$$\begin{aligned}200 - 24x =& 0 \\ x =& 200/24 = 25/3\end{aligned}$$

So, yes, there does exist one inflection point at $25/3$ other than the critical points.

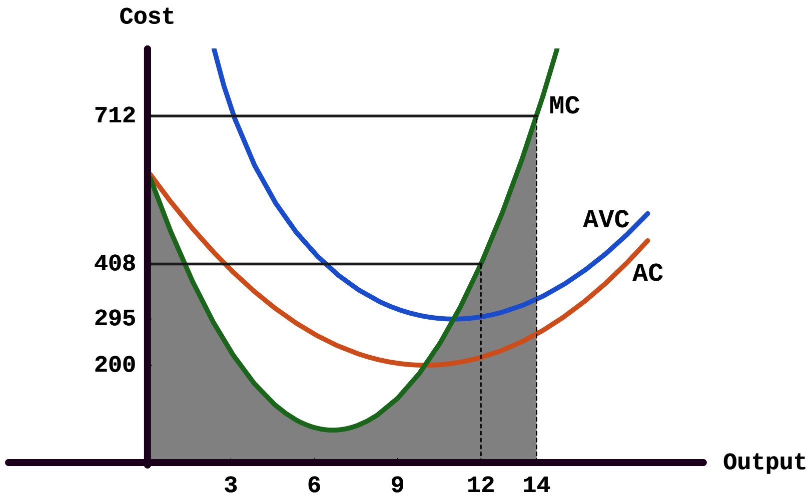
Problem 2. The cost function for a firm is a rule or mapping that tells the total cost of production of any output level produced by the firm. If the variable y represents the output of the firm, then the cost function is given by c(y). Marginal cost represents the change in the cost of production for the firm as output changes and is given by the derivative of the cost function with respect to output,i.e., Marginal Cost(MC) = $\frac{dc(y)}{dy}$. A competitive firm facing a fixed output price maximizes profit at the output level where marginal cost is equal to price as in the figure 1.

FIGURE 1. Profit Maximization



The area below the cost curve is a measure of variable cost and can be found by integrating the marginal cost curve from 0 to any given output level y. The shaded area in figure 2 represents the variable cost of production for the cost function $c(y) = 1000 + 600y - 80\,y^2 + 4y^3$.

FIGURE 2. Variable Cost of Production and Producer Surplus



Producer surplus is the area below a given price and above the marginal cost curve. Producer surplus is the unshaded area below the horizontal line at 712 in figure 2. Producer surplus can be computed by subtracting the shaded area from total revenue.

a. Find the profit maximizing level of output for the following firm. Demonstrate that the level you choose maximizes profit.

$$price \ = \ p \ = \$712$$

$$cost \ = \ c(y) \ = 1000 \ + \ 600y \ - \ 80y^2 \ + \ 4y^3$$

$$\begin{aligned}\pi =& py - c(y)\\ =& 712y - (1000 \ + \ 600y \ - \ 80y^2 \ + \ 4y^3)\\ =& -1000 + 112y + 80y^2 - 4y^3\end{aligned}$$

Letting $\frac{\partial \pi}{\partial y} = 0$, we get

$$\begin{aligned}\frac{\partial \pi}{\partial y} =& 0\\ \rightarrow 112 + 160y - 12y^2 =& 0\\ \rightarrow 12y^2 - 160y - 112 =& 0\\ \rightarrow 3y^2 - 40y - 28 =& 0\\ \rightarrow y_1 =& \frac{20 + \sqrt{484}}{3}\\ =& 42/3\\ \text{or } y_2 =& \frac{20 - \sqrt{484}}{3}\\ =& -2/3\end{aligned}$$

It is clear that $y_2 < 0$, hence the solution should be y_1. Next, let's check if y_1 maximize the profit.

$$\begin{aligned}\frac{\partial^2 \pi}{\partial y^2}|_{y_1} =& 160 - 24y\\ =& -156 < 0\end{aligned}$$

Yes, y_1 does maximize the firm's profit.

b. Find the profit maximizing level of output when the price is $408. Demonstrate that the level you choose maximizes profit.

Repeating the same steps, we have

$$\begin{aligned}
\pi =& 408y - (1000 \ + \ 600y \ - \ 80y^2 \ + \ 4y^3) \\
=& -1000 - 192y + 80y^2 - 4y^3 \\
\frac{\partial \pi}{\partial y} =& 0 \\
\rightarrow -192 + 160y - 12y^2 =& 0 \\
\rightarrow 3y^2 - 40y + 48 =& 0 \\
\rightarrow y_1 =& \frac{40 + \sqrt{40^2 - 3 \cdot 4 \cdot 48}}{6} \\
=& 36/3 \\
\text{or } y_2 =& \frac{40 - \sqrt{40^2 - 3 \cdot 4 \cdot 48}}{6} \\
=& 4/3
\end{aligned}$$

Checking the second derivative, we have

$$\begin{aligned}
\frac{\partial^2 \pi}{\partial y^2}|_{y_1} =& 160 - 24y \\
=& 160 - 24 \cdot 36/3 \\
=& -128 < 0 \\
\frac{\partial^2 \pi}{\partial y^2}|_{y_2} =& 160 - 24y \\
=& 160 - 24 \cdot 4/3 \\
=& 128 > 0
\end{aligned}$$

Hence, it is $36/3$ that maximizes the firm's profit.

c. What is variable cost for this firm when price is \$712?

$$\begin{aligned}\text{Variable cost } &= 600y \ - \ 80y^2 \ + \ 4y^3 \\ &= 600 \cdot (42/3) \ - \ 80 \cdot (42/3)^2 \ + \ 4 \cdot (42/3)^3 \\ &= 3696\end{aligned}$$

d. What is producer surplus for this firm when the price is \$712?

$$\begin{aligned}\text{Producer surplus } &= \int_0^y [p - MC(y)]dy \\ &= \int_0^{42/3} [712 - (600 - 160y + 12y^2)]dy \\ &= \int_0^{42/3} [112 + 160y - 12y^2]dy \\ &= (112y + 80y^2 - 4y^3)|_{42/3} \\ &= 8848\end{aligned}$$

e. What is variable cost for this firm when price is \$408?

$$\begin{aligned}\text{Variable cost } &= 600y \ - \ 80y^2 \ + \ 4y^3 \\ &= 600 \cdot (36/3) \ - \ 80 \cdot (36/3)^2 \ + \ 4 \cdot (36/3)^3 \\ &= 2592\end{aligned}$$

f. What is producer surplus for this firm when the price is \$408?

$$\begin{aligned}\text{Producer surplus } &= \int_0^y [p - MC(y)]dy \\ &= \int_0^{36/3} [408 - (600 - 160y + 12y^2)]dy \\ &= \int_0^{36/3} [-192 + 160y - 12y^2]dy \\ &= (-192y + 80y^2 - 4y^3)|_{36/3} \\ &= 2304\end{aligned}$$

g. How much is the firm worse off when price falls from \$712 to \$408?
The firm loses by $8848 - 2304$, i.e., 6544.

h. Cross-hatch the change in producer surplus in Figure 2.

Problem 3. For each of the following problems, write an equation that represents profit as a function of the two inputs x_1 and x_2. Write it in the form π = $pf(x_1,x_2)-w_1x_1-w_2x_2$ and then simplify the expression. Then find all first and second partial derivatives of the function.

a.

$$\begin{aligned} f(x_1, x_2) &= 5x_1 + 4x_2 - 2x_1^2 + x_1x_2 - x_2^2 \\ p &= 2 \\ w_1 &= 6, \qquad w_2 = 2 \end{aligned}$$

$$\begin{aligned} \pi =& 2(5x_1 + 4x_2 - 2x_1^2 + x_1x_2 - x_2^2) - 6x_1 - 4x_2 \\ =& 4x_1 + 6x_2 - 4x_1^2 + 2x_1x_2 - 2x_2^2 \end{aligned}$$

$\frac{\partial \pi}{\partial x_1} = 4 - 8x_1 + 2x_2$	$\frac{\partial \pi}{\partial x_2} = 6 + 2x_1 - 4x_2$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -8$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 2$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 2$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -4$

Find potential profit maximizing levels of x_1 and x_2.

Setting $\frac{\partial \pi}{\partial x_1}$ and $\frac{\partial \pi}{\partial x_2}$ to 0, we have

$$\begin{aligned} 4 - 8x_1 + 2x_2 &= 0 \\ 6 + 2x_1 - 4x_2 &= 0 \end{aligned}$$

Multiplying the first equation by 2 and adding it to the second equation, we get

$$\begin{aligned} (8 - 16x_1 + 4x_2) + 6 + 2x_1 - 4x_2 &= 0 \\ 14 - 14x_1 &= 0 \\ 14 &= 14x_1 \\ x_1 &= 1 \end{aligned}$$

Substituting $x_1 = 1$ into the second equation, we get

$$\begin{aligned} 6 + 2x_1 - 4x_2 &= 0 \\ 6 + 2 - 4x_2 &= 0 \\ 8 - 4x_2 &= 0 \\ 8 &= 4x_2 \\ x_2 &= 2 \end{aligned}$$

By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

$$H = \begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1^2} & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \pi}{\partial x_1 \partial x_2} & \frac{\partial^2 \pi}{\partial x_2^2} \end{vmatrix} = 32 - 4 = 28 > 0.$$

Since $\frac{\partial^2 \pi}{\partial x_1^2} = -8 < 0$ and $|H| > 0$, $(1, 2)$ is the maximum point.

b.

$$
\begin{aligned}
f(x_1, x_2) &= 20x_1 + 20x_2 - 2x_1^2 + 2x_1x_2 - 2x_2^2 \\
p &= 5 \\
w_1 &= 10, \qquad w_2 \ = \ 100
\end{aligned}
$$

$$
\begin{aligned}
\pi =& 5(20x_1 + 20x_2 - 2x_1^2 + 2x_1x_2 - 2x_2^2) - 10x_1 - 100x_2 \\
=& 90x_1 - 10x_1^2 + 10x_1x_2 - 10x_2^2
\end{aligned}
$$

$\frac{\partial \pi}{\partial x_1} = 90 - 20x_1 + 10x_2$	$\frac{\partial \pi}{\partial x_2} = 10x_1 - 20x_2$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -20$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 10$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 10$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -20$

Find potential profit maximizing levels of x_1 and x_2.

Setting $\frac{\partial \pi}{\partial x_1}$ and $\frac{\partial \pi}{\partial x_2}$ to 0, we have

$$\begin{aligned} 90 - 20x_1 + 10x_2 =& 0 \\ 10x_1 - 20x_2 =& 0 \end{aligned}$$

Solving these two equations, we get $x_1 = 6$ and $x_2 = 3$.

By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

$$H = \begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1 \partial x_1} & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \pi}{\partial x_1 \partial x_2} & \frac{\partial^2 \pi}{\partial x_2 \partial x_2} \end{vmatrix} = 400 - 100 = 300 > 0.$$

Since $\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -20 < 0$ and $|H| > 0$, $(6, 3)$ is the maximum point.

c.

$$
\begin{aligned}
f(x_1, x_2) &= x_1^{2/7} x_2^{1/3} \\
p &= 3024 \\
w_1 &= 81, \qquad w_2 \;=\; 448
\end{aligned}
$$

$$\pi = 3024 x_1^{2/7} x_2^{1/3} - 81 x_1 - 448 x_2$$

$\frac{\partial \pi}{\partial x_1} = 864 x_1^{-5/7} x_2^{1/3} - 81$	$\frac{\partial \pi}{\partial x_2} = 1008 x_1^{2/7} x_2^{-2/3} - 448$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = \frac{-4320}{7} x_1^{-12/7} x_2^{1/3}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 288 x_1^{-5/7} x_2^{-2/3}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 288 x_1^{-5/7} x_2^{-2/3}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -672 x_1^{2/7} x_2^{-5/3}$

Find potential profit maximizing levels of x_1 and x_2.

Setting $\frac{\partial \pi}{\partial x_1}$ and $\frac{\partial \pi}{\partial x_2}$ to 0, we have

$$\begin{aligned} 864x_1^{-5/7}x_2^{1/3} - 81 =& 0 \\ 1008x_1^{2/7}x_2^{-2/3} - 448 =& 0 \end{aligned}$$

Moving the second term to the right hand side, and dividing both equations, we have

$$\begin{aligned} \frac{864x_1^{-5/7}x_2^{1/3}}{1008x_1^{2/7}x_2^{-2/3}} =& \frac{81}{448} \\ \rightarrow \frac{864x_2}{1008x_1} =& \frac{81}{448} \\ \rightarrow \frac{x_2}{x_1} =& \frac{81 \cdot 1008}{448 \cdot 864} \\ \rightarrow \frac{x_2}{x_1} =& \frac{27}{128} \\ \rightarrow \frac{x_2}{x_1} =& \frac{3^3}{2^7} \\ \rightarrow x_2 =& \frac{3^3}{2^7}x_1 \end{aligned}$$

substituting the last equation to the first equation, we have

$$\begin{aligned}
864x_1^{-5/7}x_2^{1/3} - 81 =& 0\\
864x_1^{-5/7}(\frac{3^3}{2^7}x_1)^{1/3} - 81 =& 0\\
864x_1^{-5/7}3(2^{-7})^{1/3}x_1^{1/3} =& 81\\
864x_1^{-5/7+1/3}(2^{-7})^{1/3} =& 27\\
864x_1^{-8/21}2^{-7/3} =& 27\\
\frac{864}{27}2^{-7/3} =& x_1^{8/21}\\
32(2^{-7/3}) =& x_1^{8/21}\\
2^5(2^{-7/3}) =& x_1^{8/21}\\
2^{5-7/3} =& x_1^{8/21}\\
2^{8/3} =& x_1^{8/21}\\
(2^{8/3})^{21/8} =& (x_1^{8/21})^{21/8}\\
2^7 =& x_1\\
x_1 = 128&
\end{aligned}$$

We then easily solve out that $x_2 = 27$.

By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

$$H = \begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1 \partial x_1} & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \pi}{\partial x_1 \partial x_2} & \frac{\partial^2 \pi}{\partial x_2 \partial x_2} \end{vmatrix}$$

$\rightarrow$

$$= \begin{vmatrix} \frac{-4320}{7} x_1^{-12/7} x_2^{1/3} & 288 x_1^{-5/7} x_2^{-2/3} \\ 288 x_1^{-5/7} x_2^{-2/3} & -672 x_1^{2/7} x_2^{-5/3} \end{vmatrix}$$

$\rightarrow$

$$= \begin{vmatrix} \frac{-4320}{7} 2^{-12} 3 & 288(2^{-5})(3^{-2}) \\ 288(2^{-5})(3^{-2}) & -672(2^{2})(3^{-5}) \end{vmatrix}$$

$\rightarrow$

$$\begin{aligned} =& \frac{4320}{7} 2^{-12} 3 \cdot 672(2^2)(3^{-5}) - (288(2^{-5})(3^{-2}))^2 \\ =& 414720(2^{-10})(3^{-4}) - 82944(2^{-10})(3^{-4}) \\ =& 331776(2^{-10})(3^{-4}) \\ =& 4 > 0 \end{aligned}$$

Since $\frac{\partial^2 \pi}{\partial x_1 \partial x_1} < 0$ and $|H| > 0$, $(128, 27)$ is the maximum point.

d.

$$
\begin{aligned}
f(x_1, x_2) &= x_1^{1/6} x_2^{1/4} \\
p &= 972 \\
w_1 &= 2, \qquad w_2 \ = \ 27
\end{aligned}
$$

$$\pi = 972 x_1^{1/6} x_2^{1/4} - 2x_1 - 27x_2$$

$\frac{\partial \pi}{\partial x_1} = 162 x_1^{-5/6} x_2^{1/4} - 2$	$\frac{\partial \pi}{\partial x_2} = 243 x_1^{1/6} x_2^{-3/4} - 27$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -135 x_1^{-11/6} x_2^{1/4}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = \frac{81}{2} x_1^{-5/6} x_2^{-3/4}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 81/2 x_1^{-5/6} x_2^{-3/4}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -\frac{729}{4} x_1^{1/6} x_2^{-7/4}$

Find potential profit maximizing levels of x_1 and x_2.

Setting $\frac{\partial \pi}{\partial x_1}$ and $\frac{\partial \pi}{\partial x_2}$ to 0, we have

$$\begin{aligned} 162x_1^{-5/6}x_2^{1/4} - 2 =& 0 \\ 243x_1^{1/6}x_2^{-3/4} - 27 =& 0 \end{aligned}$$

Moving the second term to the right hand side, and dividing both equations, we have

$$\begin{aligned} \frac{162x_1^{-5/6}x_2^{1/4}}{243x_1^{1/6}x_2^{-3/4}} =& \frac{2}{27} \\ \rightarrow \frac{162x_2}{243x_1} =& \frac{2}{27} \\ \rightarrow \frac{x_2}{x_1} =& \frac{2 \cdot 243}{27 \cdot 162} \\ \rightarrow \frac{x_2}{x_1} =& \frac{1}{9} \\ \rightarrow x_2 =& \frac{1}{9}x_1 \end{aligned}$$

substituting the last equation to the first equation, we have

$$\begin{aligned} 162x_1^{-5/6}x_2^{1/4} - 2 =& 0 \\ 162x_1^{-5/6}(\frac{1}{9}x_1)^{1/4} - 2 =& 0 \\ 81x_1^{-5/6}(\frac{1}{9})^{1/4}(x_1)^{1/4} =& 1 \\ 81x_1^{-5/6+1/4}3^{-1/2} =& 1 \\ 3^4x_1^{-7/12}3^{-1/2} =& 1 \\ 3^{4-1/2} =& x_1^{7/12} \\ 3^{7/2} =& x_1^{7/12} \\ (3^{7/2})^{12/7} =& (x_1^{7/12})^{12/7} \\ 3^6 =& x_1 \\ 729 =& x_1 \end{aligned}$$

We then easily solve out that $x_2 = 81$ by substituting x_1 to the previous equation.

By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

$$H = \begin{vmatrix} \frac{\partial^2\pi}{\partial x_1 \partial x_1} & \frac{\partial^2\pi}{\partial x_1 \partial x_2} \\ \frac{\partial^2\pi}{\partial x_1 \partial x_2} & \frac{\partial^2\pi}{\partial x_2 \partial x_2} \end{vmatrix}$$

$\rightarrow$

$$= \begin{vmatrix} -135(3^{-10}) & \frac{81}{2}(3^{-8}) \\ \frac{81}{2}(3^{-8}) & -\frac{729}{4}(3^{-6}) \end{vmatrix}$$

$\rightarrow$

$$\begin{aligned} =&135 \cdot \frac{729}{4}(3^{-16}) - (\frac{81}{2}(3^{-8}))^2 \\ =&\frac{98415 - 6561}{4}3^{-16} \\ =&\frac{7}{13122} > 0 \end{aligned}$$

Since $\frac{\partial^2\pi}{\partial x_1 \partial x_1} < 0$ and $|H| > 0$, $(729, 81)$ is the maximum point.

Problem 4. Find the listed partial derivatives of each of the following functions.

a. $\mathfrak{L}(x_1, x_2, \lambda) \;=\; 6x_1 + 2x_2 - \lambda\left(5x_1 + 4x_2 - 2x_1^2 + x_1x_2 - x_2^2 - 9\right)$

$\frac{\partial \mathfrak{L}(x_1,x_2,\lambda)}{\partial x_1} = 6 - \lambda(5 - 4x_1 + x_2)$	$\frac{\partial \mathfrak{L}(x_1,x_2,\lambda)}{\partial x_2} = 2 - \lambda(4 + x_1 - 2x_2)$	$\frac{\partial \mathfrak{L}(x_1,x_2,\lambda)}{\partial \lambda} = -(5x_1 + 4x_2 - 2x_1^2 + x_1x_2 - x_2^2 - 9)$
$\frac{\partial^2 \mathfrak{L}}{\partial x_1 \partial x_1} = 4\lambda$	$\frac{\partial^2 \mathfrak{L}}{\partial x_1 \partial x_2} = -\lambda$	- $\frac{\partial^2 \mathfrak{L}}{\partial x_1 \partial \lambda} = 5 - 4x_1 + x_2$
$\frac{\partial^2 \mathfrak{L}}{\partial x_2 \partial x_1} = -\lambda$	$\frac{\partial^2 \mathfrak{L}}{\partial x_2 \partial x_2} = 2\lambda$	- $\frac{\partial^2 \mathfrak{L}}{\partial x_2 \partial \lambda} = 4 + x_1 - 2x_2$
- $\frac{\partial^2 \mathfrak{L}}{\partial \lambda \partial x_1} = 5 - 4x_1 + x_2$	- $\frac{\partial^2 \mathfrak{L}}{\partial \lambda \partial x_2} = 4 + x_1 - 2x_2$	- $\frac{\partial^2 \mathfrak{L}}{\partial \lambda \partial \lambda} = 0$

b. $\mathcal{L}(x_1, x_2, \lambda) \ = \ 20x_1 + 20x_2 - 2x_1^2 + 2x_1x_2 - 2x_2^2 - \lambda\,(10x_1 \ + \ 100x_2 \ - \ 360)$

$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 20 - 4x_1 + 2x_2 - 10\lambda$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = 20 + 2x_1 - 4x_2 - 100\lambda$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = -(10x_1 \ + \ 100x_2 \ - \ 360)$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -4$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = 2$	-$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = 10$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = 2$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -4$	- $\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = 100$
- $\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = 10$	-$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = 100$	-$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0$

c. $\mathcal{L}(x_1, x_2, \lambda) \ = \ 81x_1 + 448x_2 - \lambda\left(x_1^{2/7}x_2^{1/3} - 12\right)$

$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 81 - \frac{7}{2}\lambda x_1^{-5/7} x_2^{1/3}$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = 448 - \frac{1}{3}\lambda x_1^{2/7} x_2^{-2/3}$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = -(x_1^{2/7} x_2^{1/3} - 12)$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = \frac{5}{2}\lambda x_1^{-12/7} x_2^{1/3}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = -\frac{7}{6}\lambda x_1^{-5/7} x_2^{-2/3}$	$-\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = \frac{7}{2} x_1^{-5/7} x_2^{1/3}$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = -\frac{7}{6}\lambda x_1^{-5/7} x_2^{-2/3}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -\frac{2}{9}\lambda x_1^{2/7} x_2^{-5/3}$	$-\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = \frac{1}{3} x_1^{2/7} x_2^{-2/3}$
$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = \frac{7}{2} x_1^{-5/7} x_2^{1/3}$	$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = \frac{1}{3} x_1^{2/7} x_2^{-2/3}$	$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0$

d. $\mathcal{L}(x_1, x_2, \lambda) \ = \ x_1^{1/6} x_2^{1/4} - \lambda\,(2x_1 + 27x_2 - 3645)$

$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = \frac{1}{6} x_1^{-5/6} x_2^{1/4} - 2\lambda$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = \frac{1}{4} x_1^{1/6} x_2^{-3/4} - 27\lambda$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = -(2x_1 + 27x_2 - 3645)$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -\frac{5}{36} x_1^{-11/6} x_2^{1/4}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = \frac{1}{24} x_1^{-5/6} x_2^{-3/4}$	- $\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = 2$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = \frac{1}{24} x_1^{-5/6} x_2^{-3/4}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -\frac{3}{16} x_1^{1/6} x_2^{-7/4}$	- $\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = 27$
- $\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = 2$	- $\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = 27$	- $\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0$

Problem 5. For each of the following problems, find the critical points. For each critical point state whether the function is at a relative maximum, relative minimum, or otherwise. Check to see if there are potential points of inflection **at points other than** critical points.

a. $f(x) = 2x^4 - \frac{8}{3}x^3 - 24x^2$.

$$\begin{aligned} f'(x) =& 8x^3 - 8x^2 - 48x \\ f''(x) =& 24x^2 - 16x - 48 \end{aligned}$$

First, we solve critical points:

$$\begin{aligned} f'(x) =& 0 \\ \rightarrow 8x^3 - 8x^2 - 48x =& 0 \\ \rightarrow x^3 - x^2 - 6x =& 0 \\ \rightarrow x(x-3)(x+2) =& 0 \\ \rightarrow x_1 =& \ 0 \\ \text{or } x_2 =& \ 3 \\ \text{or } x_3 =& \ -2 \end{aligned}$$

Next, we check if the function is at a relative maximum or minimum.

$$\begin{aligned} f''(0) =& -48 < 0 \\ f''(3) =& 120 > 0 \\ f''(-2) =& 80 > 0 \end{aligned}$$

Hence, the fuction is at a relative maximum for point 0, and relative minimum at both 3 and -2.

Finally, we check if there is any inflection point other than critical points.

$$\begin{aligned} f''(0) =& 0 \\ \rightarrow 24x^2 - 16x - 48 =& 0 \\ \rightarrow 3x^2 - 2x - 6 =& 0 \\ \rightarrow x_4 =& \frac{2 + \sqrt{2^2 + 4 * 3 * 6}}{6} \\ =& \frac{2 + \sqrt{76}}{6} \\ =& \frac{1 + \sqrt{19}}{3} \\ \text{and } x_5 =& \frac{1 - \sqrt{19}}{3} \end{aligned}$$

The inflection points are $\frac{1 \pm \sqrt{19}}{3}$.

b. $f(x) = \frac{5}{4}x^4 - \frac{20}{3}x^3 - \frac{55}{2}x^2 + 150x$

$$f'(x) = 5x^3 - 20x^2 - 55x + 150$$
$$f''(x) = 15x^2 - 40x - 55$$

First, we solve critical points:

$$\begin{aligned} f'(x) &= 0 \\ \rightarrow 5x^3 - 20x^2 - 55x + 150 &= 0 \\ \rightarrow x^3 - 4x^2 - 11x + 30 &= 0 \\ \rightarrow (x-2)(x-5)(x+3) &= 0 \\ \rightarrow x_1 &= 2 \\ \text{or } x_2 &= 5 \\ \text{or } x_3 &= -3 \end{aligned}$$

Next, we check if the function is at a relative maximum or minimum.

$$\begin{aligned} f''(2) &= -75 < 0 \\ f''(5) &= 120 > 0 \\ f''(-3) &= 200 > 0 \end{aligned}$$

Hence, the fuction is at a relative maximum for point 2, and relative minimum at both 5 and -3.

Finally, we check if there is any inflection point other than critical points.

$$\begin{aligned} f''(0) &= 0 \\ \rightarrow 15x^2 - 40x - 55 &= 0 \\ \rightarrow 3x^2 - 8x - 11 &= 0 \\ \rightarrow (3x-11)(x+1) &= 0 \\ \rightarrow x_4 &= 11/3 \\ \text{and } x_5 &= -1 \end{aligned}$$

The inflection points are $\frac{11}{3}$ and -1.