ECONOMICS 207
SPRING 2007
PROBLEM SET 14

For your information, the Hessian matrix in the profit maximization problem written as

$$\pi(x_1,\, x_2) \;=\; pf(x_1,\, x_2)\; - w_1x_1 - w_2x_2$$

is given by

$$H(\pi(x_1,\, x_2)) = \begin{bmatrix} \frac{\partial^2 \pi(x_1, x_2)}{\partial x_1 \partial x_1} & \frac{\partial^2 \pi(x_1, x_2)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \pi(x_1, x_2)}{\partial x_2 \partial x_1} & \frac{\partial^2 \pi(x_1, x_2)}{\partial x_2 \partial x_2} \end{bmatrix}$$

The bordered Hessian in the constrained optimization problem written as

$$\mathcal{L}(x_1, x_2, \lambda) \;=\; f(x_1, x_2) - \lambda g(x_1, x_2)$$

is given by

$$H_B = \begin{bmatrix} \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1 \partial x_1} & \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1 \partial x_2} & -\frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2 \partial x_1} & \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2 \partial x_2} & -\frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2 \partial \lambda} \\ -\frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda \partial x_1} & -\frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda \partial x_2} & 0 \end{bmatrix}$$

Date: 16 April 2007.

Problem 1. Given the data below, write an equation that represents profit as a function of the two inputs x_1 and x_2. Write it in the form $\pi = pf(x_1,x_2) - w_1x_1 - w_2x_2$ and then simplify the expression. Then find all first and second partial derivatives of the function.

a.

$$
\begin{aligned}
f(x_1, x_2) &= 80x_1 + 60x_2 - x_1^2 + 2x_1x_2 - 2x_2^2 \\
p &= 2 \\
w_1 &= 64, \qquad w_2 \;=\; 160
\end{aligned}
$$

$\pi =$

$\frac{\partial \pi}{\partial x_1}$	$\frac{\partial \pi}{\partial x_2}$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2}$

Show that the profit maximizing levels of x_1 and x_2 are 38 and 14.

By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

$$\begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1 \partial x_1} = & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} = \\ & \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} = & \frac{\partial^2 \pi}{\partial x_2 \partial x_2} = \\ & \end{vmatrix} =$$

Problem 2. a. Given the data below, write an equation that represents profit as a function of the two inputs x_1 and x_2. Write it in the form $\pi = pf(x_1,x_2) - w_1x_1 - w_2x_2$ and then simplify the expression. Then find all first and second partial derivatives of the function.

$$\begin{aligned} f(x_1,x_2) &= x_1^{1/6}x_2^{1/5} \\ p &= 25920 \\ w_1 &= 405, \qquad w_2 = 128 \end{aligned}$$

$\pi =$

$\frac{\partial \pi}{\partial x_1} =$	$\frac{\partial \pi}{\partial x_2} =$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} =$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} =$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} =$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} =$

b. Show that the profit maximizing levels of x_1 and x_2 are 64 and 243.

c. By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

$$\begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1 \partial x_1} = & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} = \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} = & \frac{\partial^2 \pi}{\partial x_2 \partial x_2} = \end{vmatrix} = \qquad = \frac{19}{9}$$

Problem 3. a. Find the listed partial derivatives of following function.

$$\mathcal{L}(x_1, x_2, \lambda) \;=\; 40x_1 + 40x_2 - x_1^2 + 2x_1x_2 - 3x_2^2 - \lambda\,(100x_1 \;+\; 300x_2 \;-\; 5500)$$

$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 40 - 2x_1 + 2x_2 - 100\lambda$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = 40 + 2x_1 - 6x_2 - 300\lambda$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = -100x_1 - 300x_2 + 5500$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -2$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2}$	- $\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda}$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = 2$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2}$	- $\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda}$
- $\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = 100$	- $\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2}$	$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0$

b. Show that the three critical values of the function $\mathcal{L}(x_1, x_2, \lambda)$ are $x_1 = 25$, $x_2 = 10$, and $\lambda = \frac{1}{10}$.

c. Substitute the appropriate values of are x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is 360,000.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = & -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = \\ -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = & -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = \end{vmatrix} =$$

A positive determinant indicates a maximum, a negative determinant indicates a minimum.

Problem 4. a. Find the listed partial derivatives of following function.

$$\mathcal{L}(x_1, x_2, \lambda) \;=\; x_1^{1/3} x_2^{2/5} - \lambda\,(81x_1 + 50x_2 - 22275)$$

$\dfrac{\partial\mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = \frac{1}{3} x_1^{-2/3} x_2^{2/5} - 81\lambda$	$\dfrac{\partial\mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = \frac{2}{5} x_1^{1/3} x_2^{-3/5} - 50\lambda$	$\dfrac{\partial\mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda}$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -\dfrac{2}{9} x_1^{-5/3} x_2^{2/5}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} =$	- $\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} =$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} =$	- $\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda}$
- $\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = 81$	- $\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2}$	$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda}$

b. Show that the three critical values of the function $\mathcal{L}(x_1, x_2, \lambda)$ are $x_1 = 125$, $x_2 = 243$, and $\lambda = \frac{1}{675}$.

c. Substitute the appropriate values of are x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is $\frac{22}{5}$.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = & -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = \\ -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = & -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = \end{vmatrix} =$$

A positive determinant indicates a maximum, a negative determinant indicates a minimum.

Problem 5. a. Find the listed partial derivatives of following function.

$$\mathcal{L}(x_1, x_2, \lambda) \;=\; 10x_1 + 128x_2 - \lambda\left(x_1^{1/4}x_2^{2/5} - 16\right)$$

$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 10 - \frac{1}{4}\lambda x_1^{-3/4} x_2^{2/5}$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2}$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = -x_1^{1/4} x_2^{2/5} + 16$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = -\frac{\lambda}{10} x_1^{-3/4} x_2^{-3/5}$	$-\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = \frac{1}{4} x_1^{-3/4} x_2^{2/5}$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} =$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} =$	$-\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} =$
$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1}$	$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2}$	$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda}$

b. Show that the three critical values of the function $\mathcal{L}(x_1, x_2, \lambda)$ are $x_1 = 256$, $x_2 = 32$, and $\lambda = 640$.

c. Substitute the appropriate values of are x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is $-\frac{13}{5120}$.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = & -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = \\ \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = \\ \\ -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = & -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = \end{vmatrix} =$$

A positive determinant indicates a maximum, a negative determinant indicates a minimum.

Problem 6. a. Find the listed partial derivatives of following function.

$$\mathcal{L}(x_1, x_2, \lambda) \ = \ 60x_1 + 40x_2 - \lambda\left(10x_1 + 40x_2 - 2x_1^2 + x_1x_2 - x_2^2 - 476\right)$$

$\dfrac{\partial\mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 60 - 10\lambda + 4\lambda x_1 - \lambda x_2$	$\dfrac{\partial\mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = 40 - 40\lambda - \lambda x_1 + 2\lambda x_2$	$\dfrac{\partial\mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = -10x_1 - 40x_2 + 2x_1^2 - x_1x_2 + x_2^2 + 476$
$\frac{\partial^2\mathcal{L}}{\partial x_1 \partial x_1} = 4\lambda$	$\frac{\partial^2\mathcal{L}}{\partial x_1 \partial x_2}$	- $\frac{\partial^2\mathcal{L}}{\partial x_1 \partial \lambda} = 10 - 4x_1 + x_2$
$\frac{\partial^2\mathcal{L}}{\partial x_2 \partial x_1} = -\lambda$	$\frac{\partial^2\mathcal{L}}{\partial x_2 \partial x_2}$	- $\frac{\partial^2\mathcal{L}}{\partial x_2 \partial \lambda}$
- $\frac{\partial^2\mathcal{L}}{\partial \lambda \partial x_1}$	- $\frac{\partial^2\mathcal{L}}{\partial \lambda \partial x_2} =$	$\frac{\partial^2\mathcal{L}}{\partial \lambda \partial \lambda}$

b. Show that there are two sets of critical values of the function $\mathcal{L}(x_1, x_2, \lambda)$.

The first set is is $x_1 = \frac{92}{7}$, $x_2 = \frac{214}{7}$, and $\lambda = -5$.

The second set is $x_1 = 4$, $x_2 = 18$, and $\lambda = 5$.

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The second set of critical values is $x_1 = 4$, $x_2 = 18$, and $\lambda = 5$.

c. Substitute the first set of values for x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is 3,680.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -20 & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = 5 & -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = -12 \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -10 & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = -8 \\ -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = & -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = \end{vmatrix} =$$

A positive determinant indicates a maximum, a negative determinant indicates a minimum.

d. Substitute the second set of values for x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is -3,680.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = & -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = \\ -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = & -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = \end{vmatrix} =$$

A positive determinant indicates a maximum, a negative determinant indicates a minimum.