ECONOMICS 207
SPRING 2007
PROBLEM SET 15

For your information, the bordered Hessian in the constrained optimization problem written as

$$\mathcal{L}(x_1, x_2, \lambda) \;=\; f(x_1, x_2) - \lambda g(x_1, x_2)$$

is given by

$$H_B = \begin{bmatrix} \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1 \partial x_1} & \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1 \partial x_2} & \frac{\partial g(x_1, x_2)}{\partial x_1} \\ \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2 \partial x_1} & \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2 \partial x_2} & \frac{\partial g(x_1, x_2)}{\partial x_2} \\ \frac{\partial g(x_1, x_2)}{\partial x_1} & \frac{\partial g(x_1, x_2)}{\partial x_2} & 0 \end{bmatrix}$$

Date: 22 April 2007.

Problem 1. Consider a firm with a production function given by

$$y \;=\; f(x_1,\, x_2) \;=\; 20x_1 \;+\; 30x_2 \;-\; 2x_1^2 \;+\; x_1x_2 \;-\; x_2^2$$

The firm faces prices and a cost constraint given by

$$w_1 \;=\; 30, \qquad w_2 \;=\; 10, \qquad c_0 \;=\; 300$$

Find the maximum output the firm can produce given the cost constraint and the stated prices. Verify that this output is a maximum.

a. Set up the objective function for this problem and find all first and second partial derivatives of the function with respect to x_1 and x_2.

$$\mathcal{L}(x_1, x_2, \lambda) \;=$$

$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1}$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2}$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2}$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2}$

b. What is the derivative of the objective function in this problem with respect to λ?

c. Find the partial derivatives of the constraint equation with respect to x_1 and x_2.

$\frac{\partial g(x_1, x_2)}{\partial x_1}$	$\frac{\partial g(x_1, x_2)}{\partial x_2}$

d. Use the information from 1a and 1b to find critical values for x_1, x_2 and λ. Do this by by adding the term containing λ in the first two equations to both sides of the respective equations and then taking the ratio of these two equations to solve for x_1 in terms of x_2. Then substitute into the third equation to obtain an answer for x_2. Finally solve for λ.

e. Use the answers from part 1d and the expressions from parts 1a and 1c to fill in the bordered Hessian matrix for this problem. Then determine whether the critical values indicate a maximum or a minimum.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = & \frac{\partial g(x_1, x_2)}{\partial x_1} = \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = & \frac{\partial g(x_1, x_2)}{\partial x_2} = \\ \frac{\partial g(x_1, x_2)}{\partial x_1} = & \frac{\partial g(x_1, x_2)}{\partial x_2} = & 0 \end{vmatrix} =$$

Problem 2. Consider a consumer with a utility function given by

$$u = u(x_1, x_2) = 50x_1 + 25x_2 - 2x_1^2 + x_1x_2 - 2x_2^2$$

The consumer faces prices and an income constraint given by

$$p_1 = 50, \qquad p_2 = 40, \qquad m_0 = 470$$

Find potential levels of x_1 and x_2 to maximum utility for this consumer given the income constraint and the stated prices. Verify that these consumption levels maximize utility.

a. Set up the objective function for this problem and find all first and second partial derivatives of the function with respect to x_1 and x_2.

$$\mathcal{L}(x_1, x_2, \lambda) =$$

$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1}$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2}$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2}$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2}$

b. What is the derivative of the objective function in this problem with respect to λ?

c. Find the partial derivatives of the constraint equation with respect to x_1 and x_2.

$\frac{\partial g(x_1, x_2)}{\partial x_1}$	$\frac{\partial g(x_1, x_2)}{\partial x_2}$

d. Use the information from 2a and 2b to find critical values for x_1, x_2 and λ.

e. Use the answers from part 2d and the expressions from parts 2a and 2c to fill in the bordered Hessian matrix for this problem. Then determine whether the critical values indicate a maximum or a minimum.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = & \frac{\partial g(x_1, x_2)}{\partial x_1} = \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = & \frac{\partial g(x_1, x_2)}{\partial x_2} = \\ \frac{\partial g(x_1, x_2)}{\partial x_1} = & \frac{\partial g(x_1, x_2)}{\partial x_2} = & 0 \end{vmatrix} =$$

Problem 3. Consider a consumer with a utility function given by

$$u = u(x_1, x_2) = x_1^{3/5} x_2^{1/4}$$

The consumer faces prices and a utility target given by

$$p_1 = 64, \qquad p_2 = 405, \qquad u_0 = 54$$

Find potential levels of x_1 and x_2 to minimize the cost for this consumer to reach the target level of utility given the stated prices. Verify that these consumption levels minimize cost.

a. Set up the objective function for this problem and find all first and second partial derivatives of the function with respect to x_1 and x_2.

$$\mathcal{L}(x_1, x_2, \lambda) =$$

$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1}$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2}$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2}$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2}$

b. What is the derivative of the objective function in this problem with respect to λ?

c. Find the partial derivatives of the constraint equation with respect to x_1 and x_2.

$\frac{\partial g(x_1, x_2)}{\partial x_1}$	$\frac{\partial g(x_1, x_2)}{\partial x_2}$

d. Use the information from 3a and 3b to find critical values for x_1, x_2 and λ. Do this by by adding the term containing λ in the first two equations to both sides of the respective equations and then taking the ratio of these two equations to solve for x_1 in terms of x_2. Then substitute into the third equation to obtain an answer for x_2. Finally solve for λ.

e. Use the answers from part 3d and the expressions from parts 3a and 3c to fill in the bordered Hessian matrix for this problem. Then determine whether the critical values indicate a maximum or a minimum.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = & \frac{\partial g(x_1, x_2)}{\partial x_1} = \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = & \frac{\partial g(x_1, x_2)}{\partial x_2} = \\ \frac{\partial g(x_1, x_2)}{\partial x_1} = & \frac{\partial g(x_1, x_2)}{\partial x_2} = & 0 \end{vmatrix} =$$

Problem 4. Consider a consumer with a utility function given by

$$u = u(x_1, x_2) = x_1^{2/5} x_2^{1/4}$$

The consumer faces prices and a utility target given by

$$p_1 = 81, \qquad p_2 = 20, \qquad u_0 = 12$$

Find potential levels of x_1 and x_2 to minimize the cost for this consumer to reach the target level of utility given the stated prices. Verify that these consumption levels minimize cost.

a. Set up the objective function for this problem and find all first and second partial derivatives of the function with respect to x_1 and x_2.

$\mathcal{L}(x_1, x_2, \lambda) =$

$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1}$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2}$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2}$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2}$

b. What is the derivative of the objective function in this problem with respect to λ?

c. Find the partial derivatives of the constraint equation with respect to x_1 and x_2.

$\frac{\partial g(x_1, x_2)}{\partial x_1}$	$\frac{\partial g(x_1, x_2)}{\partial x_2}$

d. Use the information from 4a and 4b to find critical values for x_1, x_2 and λ. Do this by by adding the term containing λ in the first two equations to both sides of the respective equations and then taking the ratio of these two equations to solve for x_1 in terms of x_2. Then substitute into the third equation to obtain an answer for x_2. Finally solve for λ.

e. Use the answers from part 4d and the expressions from parts 4a and 4c to fill in the bordered Hessian matrix for this problem. Then determine whether the critical values indicate a maximum or a minimum.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = & \frac{\partial g(x_1, x_2)}{\partial x_1} = \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = & \frac{\partial g(x_1, x_2)}{\partial x_2} = \\ \frac{\partial g(x_1, x_2)}{\partial x_1} = & \frac{\partial g(x_1, x_2)}{\partial x_2} = & 0 \end{vmatrix} =$$

Problem 5. Consider a producer with a production function given by

$$f(x_1, x_2) \;=\; 100x_1 + 120x_2 - 3x_1^2 + 2x_1x_1 - x_2^2$$

The firm faces prices and an output target given by

$$w_1 \;=\; 180, \qquad w_2 \;=\; 220, \qquad y_0 \;=\; 1625$$

Find potential levels of x_1 and x_2 to minimize the cost for this producer to reach the target level of output given the stated prices. Verify that these input levels minimize cost.

a. Set up the objective function for this problem and find all first and second partial derivatives of the function with respect to x_1 and x_2.

$$\mathcal{L}(x_1, x_2, \lambda) \;=\;$$

$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1}$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2}$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2}$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2}$

b. What is the derivative of the objective function in this problem with respect to λ?

c. Find the partial derivatives of the constraint equation with respect to x_1 and x_2.

$\frac{\partial g(x_1, x_2)}{\partial x_1}$	$\frac{\partial g(x_1, x_2)}{\partial x_2}$

d. Use the information from 5a and 5b to find critical values for x_1, x_2 and λ. There will be two sets of critical values, one for each of the roots of the quadratic equation in the constraint. The first set of roots is $x_1 = 105$, $x_2 = 220$, and $\lambda = -2$. The second set of roots is $x_1 = 5$, $x_2 = 10$, and $\lambda = 2$.

Leave 180 on the left hand side of the first first order condition equation and move $\lambda\frac{\partial f}{x_1}$ to the right hand side. Leave 220 on the left hand side of the second first order condition equation and move $\lambda\frac{\partial f}{x_2}$ to the right hand side. Take the ratio of these two equations, eliminate λ and solve for x_1 in terms of x_2. Then substitute this expression in the third first order condition equation and solve for x_2. The roots should be 10 and 220.

WORKSPACE

e. Substitute the first set of values for x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is 256,800.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = & -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = \\ \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = \\ \\ -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = & -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = & 0 \end{vmatrix} =$$

A positive determinant indicates a maximum, a negative determinant indicates a minimum.

f. Substitute the second set of values for x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is -256,800.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = & -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = \\ -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = & -\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = & 0 \end{vmatrix} =$$

A positive determinant indicates a maximum, a negative determinant indicates a minimum.