ECONOMICS 207
SPRING 2007
PROBLEM SET 5—KEY

Problem 1. Solve the following equations for x.

a. $\dfrac{5x - 2}{4x - 28} = \dfrac{14}{11}$

Multiply both sides of the equation by $11(4x - 28)$, we have

$$\begin{aligned} 11(5x-2) =& 14(4x-28) \\ 55x - 22 =& 56x - 392 \\ 392 - 22 =& 56x - 55x \\ x =& 370 \end{aligned}$$

b. $10x^2 - 58x + 40 = 0$

Dividing both sides by 2,

$$\begin{aligned} 5x^2 - 29x + 20 =& 0 \\ (5x-4)(x-5) =& 0 \\ x =& \frac{4}{5}, 5 \end{aligned}$$

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c. $-6x^2 - 16x - 8 = 0$

First, divide both sides by -2,

$$\begin{aligned} 3x^2 + 8x + 4 =& 0 \\ (3x+2)(x+2) =& 0 \\ x =& -\frac{2}{3}, -2 \end{aligned}$$

d. $15x_1^{-2/3} - 10x_1^{-1/6} = 0$

We first move the second term to the right-hand side,

$$\begin{aligned} 15x_1^{-2/3} =& 10x_1^{-1/6} \\ \text{dividing both sides by 5, we have} \rightarrow 3x_1^{-2/3} =& 2x_1^{-1/6} \\ \text{then multiply both sides by } x_1^{2/3}, \ \rightarrow 3 =& 2x_1^{-1/6+2/3} \\ \rightarrow 3 =& 2x_1^{1/2} \\ \rightarrow \frac{3}{2} =& x_1^{1/2} \\ \rightarrow (\frac{3}{2})^2 =& (x_1^{1/2})^2 \\ \rightarrow x_1 =& \frac{9}{4} \end{aligned}$$

Problem 2. Solve the following systems of equations for x_1 and x_2 using the method of substitution.

a.

$$\{x_1 = 32, \; x_2 = 27\}$$

$$432x_1^{-4/5}x_2^{1/3} - 81 = 0 \tag{2a.1}$$

$$720x_1^{1/5}x_2^{-2/3} - 160 = 0 \tag{2a.2}$$

Rearrange the first equation 2a.1 to obtain

$$\begin{aligned} x_1^{-4/5}x_2^{1/3} &= \frac{81}{432} = \frac{3}{16} \\ \Rightarrow x_1^{4/5}x_1^{-4/5}x_2^{1/3} &= \frac{3}{16}x_1^{4/5} \\ \Rightarrow x_2^{1/3} &= \frac{3}{16}x_1^{4/5} \\ \Rightarrow x_2 &= \left(\frac{3}{16}\right)^3\left(x_1^{4/5}\right)^3 \\ &= \left(\frac{3}{16}\right)^3 x_1^{12/5} \end{aligned} \tag{2a.1.a}$$

Rearrange the second equation 2a.2 slightly to obtain

$$x_1^{1/5}x_2^{-2/3} = \frac{160}{720} = \frac{2}{9} \tag{2a.2$'$}$$

Now substitute x_2 from equation 2a.1.a into equation 2a.2′ to obtain

$$\begin{aligned} x_1^{1/5}\left(\left(\frac{3}{16}\right)^3 x_1^{12/5}\right)^{-2/3} &= \frac{2}{9} \\ \Rightarrow x_1^{1/5}\left(\frac{3}{16}\right)^{-2} x_1^{-8/5} &= \frac{2}{9} \\ \Rightarrow x_1^{-7/5}\left(\frac{3}{16}\right)^{-2} &= \frac{2}{9} \\ \Rightarrow x_1^{-7/5} &= \frac{2}{9}\left(\frac{3}{16}\right)^2 \\ \Rightarrow x_1 &= \left(\frac{2}{9}\left(\frac{3}{16}\right)^2\right)^{-5/7} = \left(\frac{2}{9}\right)^{-5/7}\left(\frac{3}{16}\right)^{-10/7} \\ &= 2^{-5/7}3^{10/7}3^{-10/7}2^{40/7} = 2^{35/7} = 2^5 = 32 \end{aligned} \tag{2a.2.a}$$

Now substitute x_1 from equation 2a.2.a into equation 2a.1.a to obtain

$$\begin{aligned} x_2 &= \left(\frac{3}{16}\right)^3 32^{12/5} \\ &= 3^3\,2^{-12}\,2^{12} = 3^3 = 27 \end{aligned}$$

b.

$$\{x_1 \;=\; 27,\; x_2 \;=\; 32\}$$

$$\begin{aligned} 180x_1^{-2/3}x_2^{2/5} \;-\; 80 &= 0 \\ 216x_1^{1/3}x_2^{-3/5} \;-\; 81 &= 0 \end{aligned}$$

First, we move the second term to the right-hand sides.

$$\begin{aligned} 180x_1^{-2/3}x_2^{2/5} &= 80 \\ 216x_1^{1/3}x_2^{-3/5} &= 81 \end{aligned}$$

Then we first divide both sides of the first equation by the left-side of the second equation, and we have,

$$\frac{180x_1^{-2/3}x_2^{2/5}}{216x_1^{1/3}x_2^{-3/5}} \;=\; \frac{80}{216x_1^{1/3}x_2^{-3/5}}\;.$$

But, since $216x_1^{1/3}x_2^{-3/5} = 81$, we can replace the denominator of the right-hand side by 81, that is,

$$\begin{aligned} \frac{180x_1^{-2/3}x_2^{2/5}}{216x_1^{1/3}x_2^{-3/5}} &= \frac{80}{81}\;, \\ \Rightarrow \frac{180x_2}{216x_1} &= \frac{80}{81}, \\ \Rightarrow x_2 &= \frac{32}{27}x_1. \end{aligned} \tag{2}$$

Next, substitute (2) into the first equations.

$$\begin{aligned} 180x_1^{-2/3}x_2^{2/5} &= 80 \\ \Rightarrow x_1^{-2/3}(\frac{32}{27}x_1)^{2/5} &= \frac{4}{9} \\ \Rightarrow x_1^{-2/3+2/5}(\frac{2^5}{3^3})^{2/5} &= \frac{4}{9} \\ \Rightarrow x_1^{-4/15}(2^5 3^{-3})^{2/5} &= \frac{4}{9} \\ \Rightarrow x_1^{-4/15}2^2 3^{-6/5} &= \frac{4}{9} \\ \Rightarrow x_1^{4/15} &= 3^{-6/5}9 \\ \Rightarrow x_1^{4/15} &= 3^{2-6/5} \\ \Rightarrow x_1^{4/15} &= 3^{4/5} \\ \Rightarrow x_1 &= (3^{4/5})^{15/4} \\ \Rightarrow x_1 &= 3^3 = 27 \end{aligned} \tag{3}$$

At last, substitute (3) into (2), we have

$$x_2 = 32$$

Problem 3. Solve the following systems of equation for x_1, x_2, and x_3 first using the method of substitution and then using the method of elimination.

a.

$$\{x_1 = -1,\ x_2 = 1,\ x_3 = -2\}$$

$$\begin{aligned} x_1 + 2x_2 + 5x_3 &= -9 \\ 3x_1 + 7x_2 + 16x_3 &= -28 \\ 3x_1 - x_2 + 9x_3 &= -22 \end{aligned}$$

Start from the first equation.

$$x_1 = -9 - 2x_2 - 5x_3 \tag{4}$$

Then substitute (4) into the second equation, we have

$$\begin{aligned} 3(-9 - 2x_2 - 5x_3) + 7x_2 + 16x_3 &= -28 \\ -27 - 6x_2 - 15x_3 + 7x_2 + 16x_3 &= -28 \\ x_2 + x_3 &= -1 \\ x_2 &= -1 - x_3. \end{aligned} \tag{5}$$

Next, substitute (4) and (5) into the last equation.

$$\begin{aligned} 3x_1 - x_2 + 9x_3 &= -22 \\ 3(-9 - 2x_2 - 5x_3) - x_2 + 9x_3 &= -22 \\ -27 - 6x_2 - 15x_3 - x_2 + 9x_3 &= -22 \\ -7x_2 - 6x_3 &= 5 \\ -7(-1 - x_3) - 6x_3 &= 5 \\ 7 + 7x_3 - 6x_3 &= 5 \\ x_3 &= -2 \end{aligned} \tag{6}$$

We then substitute (6) into (5) and (4), and we have

$$x_2 = -1 - (-2) = 1, \tag{7}$$

$$x_1 = -9 - 2x_2 - 5x_3 = -9 - 2 + 10 = -1 \tag{8}$$

Elimination here

$$\begin{aligned} x_1 + 2x_2 + 5x_3 &= -9 \\ 3x_1 + 7x_2 + 16x_3 &= -28 \\ 3x_1 - x_2 + 9x_3 &= -22 \end{aligned}$$

We first subtract the third equation from the second one.

$$8x_2 + 7x_3 = -6 \tag{9}$$

Then, we multiply the first one by -3 and add it to the second one.

$$x_2 + x_3 = -1 \tag{10}$$

We then multiply (10) by 7 and subtract that from (9).

$$x_2 = 1 \tag{11}$$

Then we can solve x_3 out.

$$\begin{aligned} 1 + x_3 &= -1 \\ x_3 &= -2 \end{aligned} \tag{12}$$

And substituting x_2, x_3 into the first one, we get

$$\begin{aligned} x_1 + 2x_2 + 5x_3 &= -9 \\ x_1 &= -9 - 2(1) - 5(-2) \\ x_1 &= -1 \end{aligned} \tag{13}$$

b.

$$\{x_1 = 2, \ x_2 = 1, \ x_3 = 6\}$$

$$x_1 + 3x_2 + 2x_3 = 17$$

$$2x_1 + 7x_2 + 5x_3 = 41$$

$$-4x_1 - x_2 + 4x_3 = 15$$

Start from the first equation.

$$x_1 = 17 - 3x_2 - 2x_3 \tag{14}$$

Then subsitute that into the second one.

$$\begin{aligned} 2(17 - 3x_2 - 2x_3) + 7x_2 + 5x_3 =& 41 \\ 34 - 6x_2 - 4x_3 + 7x_2 + 5x_3 =& 41 \\ x_2 + x_3 =& 7 \\ x_2 =& 7 - x_3 \end{aligned} \tag{15}$$

Next, we substitute (14) and (15) into the third one.

$$\begin{aligned} -4(17 - 3x_2 - 2x_3) - x_2 + 4x_3 =& 15 \\ -68 + 12x_2 + 8x_3 - x_2 + 4x_3 =& 15 \\ 11x_2 + 12x_3 =& 83 \\ 11(7 - x_3) + 12x_3 =& 83 \\ 77 - 11x_3 + 12x_3 =& 83 \\ x_3 =& 6 \end{aligned} \tag{16}$$

Substitute $x_3 = 6$ into (14) and (15), we get

$$x_2 = 7 - x_3 = 7 - 6 = 1$$

$$x_1 = 17 - 3x_2 - 2x_3 = 17 - 3 - 12 = 2$$

Elimination here

We first subtract the second equation by the first one multiplied by 2.

$$x_2 + x_3 = 7 \tag{17}$$

Next, we multiply the second equation by 2 and add it to the last one.

$$13x_2 + 14x_3 = 97 \tag{18}$$

Then, multiply the (17) by 13 and subtract that from the equation (18).

$$x_3 = 6 \tag{19}$$

Plug that into (17), we get

$$x_2 = 7 - x_3 = 1 \tag{20}$$

Finally, plug the solutions for x_2 and x_3 into the first equation.

$$x_1 = 17 - 3x_2 - 2x_3 = 17 - 3 - 12 = 2 \tag{21}$$

Problem 4. Do the following problems from the book.

a. Section 6.2
   1) 1 (The comparison is with equation 6 in the text, not problem 6)
   2) 3
   3) 5

b. Section 6.3
   1) 1

c. Section 6.4
   1) 1
   2) 3
   3) 7

d. Section 6.5
   1) 1a
   2) 1b
   3) 1c
   4) 1d

e. Section 6.6
   1) 3a
   2) 3b
   3) 3c
   4) 3d
   5) 3e
   6) 3f
   7) 3g
   8) 3h

Problem 5. Find the derivatives of each of the following functions with respect to x.

a. $y = 4x^2 - 3x^3$

$$\begin{aligned} y' &= 4 \cdot 2x^{2-1} - 3 \cdot 3x^{3-1} \\ &= 8x - 9x^2 \end{aligned}$$

b. $f(x) = 4\,e^x + \frac{1}{2}x^2$

$$f'(x) = 4e^x + x$$

c. $f(x) = 27x^{1/3} - 2\log[x]$

$$f'(x) = 9x^{-2/3} - 2/x$$

d. $f(x) = 3x^2 + 2x^3 - 5^x$

$$f'(x) = 6x + 6x^2 - 5^x \ln 5$$

e. $f(x) = 3x^{1/2}z^{1/3} - 4x^{-1}$

$$f'(x) = \frac{3}{2}x^{-1/2}z^{1/3} + 4x^{-2}$$

f. $f(x) \;=\; x^{-2} + 3x^2\, e^x$

$$f'(x) = -2x^{-3} + 6xe^x + 3x^2e^x$$

g. $f(x) \;=\; 2x^3 \log[x]$

$$\begin{aligned} f'(x) =& (2x^3)' \log x + 2x^3 (\log x)' \\ =& 6x^2 \log x + \frac{2x^3}{x} \\ =& 6x^2 \log x + 2x^2 \end{aligned}$$

h. $f(x) \;=\; (2x \;+\; 5)^2$ Find in two different ways.

$$\begin{aligned} f'(x) =& 2(2x+5)(2x+5)' \\ =& 2(2x+5)2 \\ =& 4(2x+5) \\ =& 8x + 20 \\ f'(x) =& (4x^2 + 20x + 25)' \\ =& 8x + 20 \end{aligned}$$

i. $f(x) \;=\; \frac{x^3+2x}{x^2+3}$

$$\begin{aligned} f'(x) =& \frac{(x^3+2x)'(x^2+3) - (x^3+2x)(x^2+3)'}{(x^2+3)^2} \\ =& \frac{(3x^2+2)(x^2+3) - (x^3+2x)2x}{(x^2+3)^2} \end{aligned}$$

j. $f(x) \;=\; \frac{2x^3\, e^x}{x^2+\log[x]}$

$$\begin{aligned} f'(x) =& \frac{(2x^3\, e^x)'(x^2 + \log[x]) - (2x^3\, e^x)(x^2 + \log[x])'}{(x^2 + \log[x])^2} \\ =& \frac{(6x^2\, e^x + 2x^3 e^x)(x^2 + \log[x]) - (2x^3\, e^x)(2x + \frac{1}{x})}{(x^2 + \log[x])^2} \end{aligned}$$

Problem 6. Find the derivatives of each of the following functions with respect to x.

a. $f(x) = (2x^2 + 3x)^3$

$$\begin{aligned} f'(x) =& 3(2x^2+3x)^2(2x^2+3x)' \\ =& 3(2x^2+3x)^2(4x+3) \end{aligned}$$

b. $f(x) = (2x + 2)(2x - 4)$ Show two ways.

$$\begin{aligned} f'(x) =& (2x + 2)'(2x - 4) + (2x + 2)(2x - 4)' \\ =& 2(2x-4) + 2(2x+2) \\ =& 8x - 4 \end{aligned}$$

$$\begin{aligned} f'(x) =& [(2x + 2)(2x - 4)]' \\ =& [4x^2 - 4x - 8]' \\ =& 8x - 4 \end{aligned}$$

c. $f(x) = 2x^2 e^{x^2+2}$

$$\begin{aligned} f'(x) =& (2x^2)'e^{x^2+2} + 2x^2(e^{x^2+2})' \\ =& 4xe^{x^2+2} + 2x^2e^{x^2+2}2x \\ =& 4xe^{x^2+2} + 4x^3e^{x^2+2} \end{aligned}$$

d. $f(x) = 5^x e^{3x^2+e^x}$

$$\begin{aligned} f'(x) =& (5^x)'e^{3x^2+e^x} + 5^x(e^{3x^2+e^x})' \\ =& 5^x \ln 5 e^{3x^2+e^x} + 5^x e^{3x^2+e^x}(6x+e^x) \end{aligned}$$

e. $f(x) \;=\; x^2\, e^{x^2-3x}$

$$\begin{aligned} f'(x) =& (x^2)' e^{x^2-3x} + x^2 (e^{x^2-3x})' \\ =& 2xe^{x^2-3x} + x^2 e^{x^2-3x}(2x-3) \end{aligned}$$

f. $f(x) \;=\; \log[(x^2+2x)^3]$

$$\begin{aligned} f'(x) =& \frac{((x^2+2x)^3)'}{(x^2+2x)^3} \\ =& \frac{3(x^2+2x)^2(2x+2)}{(x^2+2x)^3} \\ =& \frac{3(2x+2)}{x^2+2x} \end{aligned}$$

g. $f(x) \;=\; \frac{3x\, e^{2x}}{4x^{1/2}+2}$

$$\begin{aligned} f'(x) =& \frac{(3xe^{2x})'(4x^{1/2}+2) - (3xe^{2x})(4x^{1/2}+2)'}{(4x^{1/2}+2)^2} \\ =& \frac{(3e^{2x}+6xe^{2x})(4x^{1/2}+2) - (3xe^{2x})2x^{-1/2}}{(4x^{1/2}+2)^2} \\ =& \frac{(3e^{2x}+6xe^{2x})(4x^{1/2}+2) - 6x^{1/2}e^{2x}}{(4x^{1/2}+2)^2} \end{aligned}$$

h. $f(x) \;=\; \frac{3x\, e^{2x^2}}{x^2 \;+\; 2\log[x]}$

$$\begin{aligned} f'(x) =& \frac{(3xe^{2x^2})'(x^2 \;+\; 2\log[x]) - (3xe^{2x^2})(x^2 \;+\; 2\log[x])'}{(x^2 \;+\; 2\log[x])^2} \\ =& \frac{(3e^{2x^2}+12x^2e^{2x^2})(x^2 \;+\; 2\log[x]) - (3xe^{2x^2})(2x \;+\; \frac{2}{x})}{(x^2 \;+\; 2\log[x])^2} \end{aligned}$$

Problem 7. For each of the following, take the derivative with respect to x_1, set the derivative equal to zero and solve the resulting equation for x_1.

a. $f(x) = 81x_1^{1/3} - 3x_1$

$$\begin{aligned} f'(x) =& 27x_1^{-2/3} - 3 = 0 \\ \Rightarrow 27x_1^{-2/3} =& 3 \\ \Rightarrow 9 =& x_1^{2/3} \\ \Rightarrow 9^{3/2} =& x_1 \\ \Rightarrow x_1 =& 27 \end{aligned}$$

b. $f(x) = 54x_1^{2/3} - 9x_1$

$$\begin{aligned} f'(x) =& 36x_1^{-1/3} - 9 = 0 \\ \Rightarrow 36x_1^{-1/3} =& 9 \\ \Rightarrow 4 =& x_1^{1/3} \\ \Rightarrow 4^3 =& x_1 \\ \Rightarrow x_1 =& 64 \end{aligned}$$

c. $f(x) \;=\; 64x_1^{5/8} \;-\; 5x_1$

$$\begin{aligned} f'(x) &= 40x_1^{-3/8} \;-\; 5 \;=\; 0 \\ \Rightarrow\; 40x_1^{-3/8} &= 5 \\ \Rightarrow\; x_1^{-3/8} &= \frac{5}{40} \;=\; \frac{1}{8} \\ \Rightarrow\; x_1 &= \left(\frac{1}{8}\right)^{-8/3} \;=\; 8^{8/3} \;=\; 2^8 \;=\; 256 \end{aligned}$$

d. $f(x) \;=\; 432x_1^{1/4}x_2^{1/3} \;-\; 16x_1 \;-\; 27x_2$

$$\begin{aligned} f'(x_1) &= 108x_1^{-3/4}x_2^{1/3} \;-\; 16 \;=\; 0 \\ \Rightarrow\; 108x_1^{-3/4}x_2^{1/3} &= 16 \\ \Rightarrow\; 108x_1^{-3/4} &= 16x_2^{-1/3} \\ \Rightarrow\; x_1^{-3/4} &= \frac{16}{108}x_2^{-1/3} \;=\; \frac{4}{27}x_2^{-1/3} \\ \Rightarrow\; x_1 &= \left(\frac{4}{27}x_2^{-1/3}\right)^{-4/3} \;=\; \left(\frac{4}{27}\right)^{-4/3} x_2^{4/9} \\ \Rightarrow\; x_1 &= \left(\frac{27}{4}\right)^{4/3} x_2^{4/9} \;=\; 81\left(\frac{1}{4}\right)^{4/3} x_2^{4/9} \end{aligned}$$

Problem 8. For the given function find an equation of the tangent line at the specified point x = a.

a.

$$\begin{aligned} y &= 3x^2 + 2x \\ a &= 3 \end{aligned}$$

The derivative of y = f(x) = f'(x) = 6x+2. The equation for the tangent to a curve f(·) at the point a is given by

$$\begin{aligned} tangent(x) &= f(a) + f'(a)(x-a) \\ &= f(a) - f'(a) + f'(a)x \end{aligned}$$

Evaluating f(x) and f'(x) at x=3 will give

$$\begin{aligned} f(3) &= (3)(3^2) + (2)(3) = 27 + 6 = 33 \\ f'(3) &= (6)(3) + 2 = 20 \end{aligned}$$

Substituting we then obtain

$$\begin{aligned} tangent(x) &= 33 - 20(3) + 20x \\ &= -27 + 20x \end{aligned}$$

b.

$$\begin{aligned} y &= 10x + 2x^2 - 0.01x^3 \\ a &= 10 \end{aligned}$$

The derivative of $y = f(x) = f'(x) = 10 + 4x - 0.03x^2$. The equation for the tangent to a curve f(·) at the point a is given by

$$\begin{aligned} tangent(x) &= f(a) + f'(a)(x-a) \\ &= f(a) - f'(a) + f'(a)x \end{aligned}$$

Evaluating f(x) and f'(x) at x=10 will give

$$\begin{aligned} f(10) &= 100 + 200 - 10 = 290 \\ f'(10) &= 10 + 40 - 3 = 47 \end{aligned}$$

Substituting we then obtain

$$\begin{aligned} tangent(x) &= 290 + 47(x-10) \\ &= -180 + 47x \end{aligned}$$