ECONOMICS 207
SPRING 2007
PROBLEM SET 6—KEY

Problem 1. Find the derivatives of each of the following functions with respect to x.

a. $y = \frac{3x^2}{(2x^3+2)^2}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(3x^2)'(2x^3+2)^2 - (3x^2)((2x^3+2)^2)'}{(2x^3+2)^4} \\ &= \frac{6x(2x^3+2)^2 - (3x^2)(2(2x^3+2)(6x^2))}{(2x^3+2)^4} \\ &= \frac{6x(2x^3+2) - (3x^2)(2(6x^2))}{(2x^3+2)^3} \\ &= \frac{12x^4 + 12x - 36x^4}{(2x^3+2)^3} \\ &= \frac{12x - 24x^4}{(2x^3+2)^3}\end{aligned}$$

b. $y = \frac{e^{2x^3}}{2x^2e^{2x}}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{(e^{2x^3})'2x^2e^{2x} - e^{2x^3}(2x^2e^{2x})'}{(2x^2e^{2x})^2} \\ &= \frac{e^{2x^3}6x^2 2x^2e^{2x} - e^{2x^3}(4xe^{2x} + 2x^2e^{2x}2)}{4x^4e^{4x}} \\ &= \frac{12x^4e^{2x^3+2x} - e^{2x^3}(4xe^{2x} + 4x^2e^{2x})}{4x^4e^{4x}} \\ &= \frac{12x^4e^{2x^3+2x} - e^{2x^3+2x}(4x + 4x^2)}{4x^4e^{4x}} \\ &= \frac{3x^3e^{2x^3} - e^{2x^3}(1+x)}{x^3e^{2x}}\end{aligned}$$

Date: 18 January 2008.

c. Find the derivative with respect to x_1. $y = 180x_1^{1/2}x_2^{2/5} - 40x_1 - 81x_2$

$$\begin{aligned}\frac{dy}{dx_1} &= 180 \cdot \frac{1}{2}x_1^{1/2-1}x_2^{2/5} - 40 \\ &= 90x_1^{-1/2}x_2 2/5 - 40\end{aligned}$$

d. Find the derivative with respect to x_2. $y = 180x_1^{1/2}x_2^{2/5} - 40x_1 - 81x_2$

$$\begin{aligned}\frac{dy}{dx_2} &= 180 \cdot \frac{2}{5}x_1^{1/2}x_2^{2/5-1} - 81 \\ &= 72x_1^{1/2}x_2 - 3/5 - 81\end{aligned}$$

Problem 2. Find the second derivative of each of the following functions with respect to x

a. $y = 6x^{1/3} - 2x^{1/2} + 2x^2$

$$\begin{aligned}\frac{dy}{dx} =& 6 \cdot \frac{1}{3}x^{1/3-1} - 2 \cdot \frac{1}{2}x^{1/2-1} + 2 \cdot 2x^{2-1}\\ =& 2x^{-2/3} - x^{-1/2} + 4x\\ \frac{d^2y}{dx^2} =& 2 \cdot (-2/3)x^{-2/3-1} - (-1/2)x^{-1/2-1} + 4\\ =& -\frac{4}{3}x^{-5/3} + \frac{1}{2}x^{-3/2} + 4\end{aligned}$$

b. $y = \frac{3x^3}{(2x^2 + 5)^2}$

$$\begin{aligned}y' =& \frac{(3x^3)'(2x^2 + 5)^2 - 3x^3((2x^2 + 5)^2)'}{(2x^2 + 5)^4}\\ =& \frac{9x^2(2x^2 + 5)^2 - 3x^3 2(2x^2 + 5)(4x)}{(2x^2 + 5)^4}\\ =& \frac{9x^2(2x^2 + 5) - 24x^4}{(2x^2 + 5)^3}\\ =& \frac{45x^2 - 6x^4}{(2x^2 + 5)^3}\\ y'' =& \frac{(45x^2 - 6x^4)'(2x^2 + 5)^3 - (45x^2 - 6x^4)((2x^2 + 5)^3)'}{(2x^2 + 5)^6}\\ =& \frac{(90x - 24x^3)(2x^2 + 5)^3 - (45x^2 - 6x^4)3(2x^2 + 5)^2(4x)}{(2x^2 + 5)^6}\\ =& \frac{(90x - 24x^3)(2x^2 + 5) - 12x(45x^2 - 6x^4)}{(2x^2 + 5)^6}\\ =& \frac{450x + 24x^5 - 480x^3}{(2x^2 + 5)^4}\end{aligned}$$

c. $y = ln[(3x^2 - 5x)^2]$

$$\begin{aligned}
y' &= \frac{[(3x^2 - 5x)^2]'}{(3x^2 - 5x)^2} \\
&= \frac{2(3x^2 - 5x)(3x^2 - 5x)'}{(3x^2 - 5x)^2} \\
&= \frac{2(6x - 5)}{3x^2 - 5x} \\
&= \frac{12x - 10}{3x^2 - 5x} \\
y'' &= \frac{(12x - 10)'(3x^2 - 5x) - (12x - 10)(3x^2 - 5x)'}{(3x^2 - 5x)^2} \\
&= \frac{12(3x^2 - 5x) - (12x - 10)(6x - 5)}{(3x^2 - 5x)^2} \\
&= \frac{36x^2 - 60x - 72x^2 + 120x - 50}{(3x^2 - 5x)^2} \\
&= \frac{-36x^2 + 60x - 50}{(3x^2 - 5x)^2}
\end{aligned}$$

d. $y = 5(200x + 30x^2 - x^3) - 2125x$

$$\begin{aligned}
y' &= 5(200 + 60x - 3x^2) - 2125 \\
y'' &= 5(60 - 6x) \\
&= 300 - 30x
\end{aligned}$$

Problem 3. In the following problems you are given a production function for a firm where y is the level of output and x is the level of the variable input. You are given the price (p)of the output and the price (w) of the single variable input. For each problem write down an equation that represents profit for the firm. Then maximize this function by taking its derivative with respect to the variable input x and set equal to zero. What is the optimal level of x, of output, of cost, of revenue, of profit?

a.

$$output\ price\ =\ p\ = 5$$

$$input\ price\ =\ w\ = 2125$$

$$y\ =\ output\ =\ f(x)\ =\ 200x\ +\ 30x^2\ -x^3$$

Solution:

The profit function is

$$\begin{aligned}\pi =& py - wx\\ =& 5(200x\ +\ 30x^2\ -x^3) - 2125x\\ =& 1000x + 150x^2 - 5x^3 - 2125x\\ =& 150x^2 - 5x^3 - 1125x\end{aligned}$$

We first set the first derivative of the profit function to 0, that is,

$$\begin{aligned}\pi' =& 0\\ \rightarrow 300x - 15x^2 - 1125 =& 0\\ \rightarrow 15x^2 - 300x + 1125 =& 0\\ \rightarrow x^2 - 20x + 75 =& 0\\ \rightarrow (x-5)(x-15) =& 0\\ \rightarrow x =& 5, 15\end{aligned}$$

To check which of the solution is the optimal level of x, we use the second derivative:

$$\begin{aligned}\pi'' =& 300 - 30x\\ \rightarrow \pi''(5) =& 300 - 150\\ =& 150 > 0\\ \pi''(15) =& 300 - 450\\ =& -150 < 0\end{aligned}$$

Since $\pi''(15) < 0$, 15 is the optimal level of x.

And the optimal levle of cost is $wx = 2125 \times 15 = 31875$. The revenue is $py = 5(200 \cdot 15 + 30 \cdot 15^2 - 15^3) = 31875$ and the profit is $py - wx = 0$.

b.

$$output\ price\ =\ p\ = 3$$

$$input\ price\ =\ w\ = 1872$$

$$y\ =\ output\ =\ f(x)\ =\ 400x\ +\ 50x^2\ - 2x^3$$

Solution:

The profit function is

$$\begin{aligned}\pi =& py - wx \\ =& 3(400x\ +\ 50x^2\ - 2x^3) - 1872x \\ =& 1200x + 150x^2 - 6x^3 - 1872x \\ =& 150x^2 - 6x^3 - 672x\end{aligned}$$

We first set the first derivative of the profit function to 0, that is,

$$\begin{aligned}\pi' =& 0 \\ \rightarrow 300x - 18x^2 - 672 =& 0 \\ \rightarrow 18x^2 - 300x + 672 =& 0 \\ \rightarrow 3x^2 - 50x + 112 =& 0 \\ \rightarrow x =& \frac{50 \pm \sqrt{2500 - 3 \cdot 4 \cdot 112}}{6} \\ x =& \frac{25 \pm \sqrt{289}}{3}\end{aligned}$$

To check which of the solution is the optimal level of x, we use the second derivative:

$$\begin{aligned}\pi'' =& 300 - 36x \\ \rightarrow \pi''(\frac{25 + \sqrt{289}}{3}) =& 300 - 12(25 + \sqrt{289}) \\ =& 300 - 300 - 12\sqrt{289} \\ =& -12\sqrt{289} < 0 \\ \pi''(\frac{25 - \sqrt{289}}{3}) =& 300 - 12(25 - \sqrt{289}) \\ =& 300 - 300 + 12\sqrt{289} \\ =& 12\sqrt{289} > 0\end{aligned}$$

Since $\pi''(\frac{25+\sqrt{289}}{3}) < 0$, $\frac{25+\sqrt{289}}{3}$ is the optimal level of x.

Problem 4. For each of the following problems you are given a price (p) and the cost function for a competitive firm. TC(y) stands for total cost and y represents the level of output. Marginal cost is the derivative of the cost function with respect to output. Find the profit maximizing level of output for each case.

a.

$$price \;=\; p \;= 392$$

$$Total\ Cost \;=\; TC \;= 500 \;+\; 200y \;-\; 10y^2 \;+\; y^3$$

The profit function is

$$\begin{aligned}\pi =& py - TC\\ =& 392y - 500 - 200y + 10y^2 - y^3\\ =& -500 + 192y + 10y^2 - y^3\end{aligned}$$

We find the first derivative and set it to 0,

$$\begin{aligned}\pi' =& 0\\ \to 192 + 20y - 3y^2 =& 0\\ \to 3y^2 - 20y - 192 =& 0\\ \to (y-12)(3y+16) =& 0\\ \to y =& 12, -\frac{16}{3}.\end{aligned}$$

Next, we check the second derivative at the above solutions,

$$\begin{aligned}\pi'' =& 20 - 6y\\ \to \pi''(12) =& 20 - 72\\ =& -52 < 0,\\ \pi''(-16/3) =& 20 + 32\\ =& 52 > 0.\end{aligned}$$

Hence, 12 is the optimal level of output.

b.

$$p = 496$$
$$TC = 1000 + 400y - 50y^2 + 3y^3$$

The profit function is

$$\begin{aligned}\pi =& py - TC\\ =& 496y - 1000 - 400y + 50y^2 - 3y^3\\ =& -1000 + 96y + 50y^2 - 3y^3\end{aligned}$$

We find the first derivative and set it to 0,

$$\begin{aligned}\pi' =& 0\\ \rightarrow 96 + 100y - 9y^2 =& 0\\ \rightarrow 9y^2 - 100y - 96 =& 0\\ \rightarrow y =& \frac{100 \pm \sqrt{10000 + 4 \cdot 9 \cdot 96}}{18}\\ y =& \frac{50 \pm \sqrt{3364}}{9}.\end{aligned}$$

Next, we check the second derivative at the above solutions,

$$\begin{aligned}\pi'' =& 100 - 18y\\ \rightarrow \pi''(\frac{50 + \sqrt{3364}}{9}) =& 100 - 2(50 + \sqrt{3364})\\ =& -2\sqrt{3364} < 0,\\ \pi''(\frac{50 - \sqrt{3364}}{9}) =& 100 - 2(50 - \sqrt{3364})\\ =& 2\sqrt{3364} > 0.\end{aligned}$$

Hence, $\frac{50+\sqrt{3364}}{9}$ is the optimal level of output.

Problem 5. Solve the following systems of equations.

$$90x_1^{-1/2}x_2^{2/5} - 40 = 0$$
$$72x_1^{1/2}x_2^{-3/5} - 81 = 0$$

Solution:
Step 1: move the second term of two equations to the right-hand side.

$$90x_1^{-1/2}x_2^{2/5} = 40,$$
$$72x_1^{1/2}x_2^{-3/5} = 81.$$

Step2: divide two equations:

$$\begin{aligned}
\frac{90x_1^{-1/2}x_2^{2/5}}{72x_1^{1/2}x_2^{-3/5}} &= \frac{40}{81}, \\
\to \frac{90x_2}{72x_1} &= \frac{40}{81} \\
\to \frac{x_2}{x_1} &= \frac{40}{81}\frac{72}{90} \qquad (1) \\
\to \frac{x_2}{x_1} &= \frac{32}{81} \\
\to x_2 &= \frac{32}{81}x_1
\end{aligned}$$

Step3: we substitute equation (1) into the first equation,

$$\begin{aligned}
90x_1^{-1/2}(\frac{32}{81}x_1)^{2/5} &= 40 \\
\to 9x_1^{-1/2}(\frac{32}{81}x_1)^{2/5} &= 4 \\
\to 9x_1^{-1/2}(\frac{32}{81})^{2/5}x_1^{2/5} &= 4 \\
\to x_1^{-1/2+2/5}9(2^5)^{2/5}(3^{-4})^{2/5} &= 4 \\
\to x_1^{-1/10}92^2 3^{-8/5} &= 4 \\
\to x_1^{-1/10}3^2 3^{-8/5} &= 1 \qquad (2) \\
\to x_1^{-1/10}3^{2-8/5} &= 1 \\
\to x_1^{-1/10}3^{2/5} &= 1 \\
\to x_1^{1/10} &= 3^{2/5} \\
\to x_1 &= (3^{2/5})^{10} \\
\to x_1 &= 3^4 = 81 \\
\to x_2 &= 32.
\end{aligned}$$

Problem 6. Find the indefinite integral of each of the following functions. Write in the form F(x) + c.

a. $f(x) = 5x^2 + 12x + 4$

$$\begin{aligned}\int f(x)dx =& \frac{5}{2+1}x^{2+1} + \frac{12}{2}x^{1+1} + 4x + c \\ =& \frac{5}{3}x^3 + 6x^2 + 4x + c\end{aligned}$$

b. $f(x) = 8x^{-1/3} - 4$

$$\begin{aligned}\int f(x)dx =& \frac{8}{-1/3+1}x^{-1/3+1} - 4x + c \\ =& \frac{8}{2/3}x^{2/3} - 4x + c \\ =& 12x^{2/3} - 4x + c\end{aligned}$$

c. $y = \frac{1}{x}$

$$\int ydx = lnx + c$$

d. $y = 6x^{-2} - 60x^{-1/2} + 200$

$$\begin{aligned}\int f(x)dx =& \frac{6}{-2+1}x^{-2+1} - \frac{60}{-1/2+1}x^{-1/2+1} + 200x + c \\ =& -\frac{6}{x} - 120x^{1/2} + 200x + c\end{aligned}$$

Problem 7. Find the definite integral of each of the following functions.

a. $\int_1^3 (3x + 4)\, dx$

$$\begin{aligned}\int_1^3 (3x + 4)\, dx =& (\frac{3}{1+1}x^{1+1} + 4x + c)|_1^3 \\ =& (\frac{3}{2}x^2 + 4x + c)|_1^3 \\ =& \frac{3}{2}3^2 + 12 + c - \frac{3}{2} - 4 - c \\ =& \frac{27}{2} + 8 - \frac{3}{2} \\ =& 20\end{aligned}$$

b. $\int_0^{15} \left(3x^2 - 24x + 200\right) dx$

$$\begin{aligned}\int_0^{15} \left(3x^2 - 24x + 200\right) dx =& (\frac{3}{2+1}x^{2+1} - \frac{24}{2}x^2 + 200x + c)|_0^{15} \\ =& (x^3 - 12x^2 + 200x + c)|_0^{15} \\ =& 15^3 - 12 \times 15^2 + 200 \times 15 + c - c \\ =& 3675\end{aligned}$$

c. $\int_4^{12} \left(9x^2 \; - \; 60x \; + \; 400\right) \, dx$

$$\begin{aligned}\int_4^{12} \left(9x^2 \; - \; 60x \; + \; 400\right) \, dx =& (3x^3 - 30x^2 + 400x + c)|_4^{12} \\ =& (3(12)^3 - 30(12)^2 + 400(12) + c) - (34^3 - 304^2 + 400(4) + c) \\ =& 5184 - 4320 + 4800 - 192 + 480 - 1600 \\ =& 4352\end{aligned}$$

d. $\int_0^{11} \left(6x^2 \; - \; 30x \; + \; 400\right) \, dx$

$$\begin{aligned}\int_0^{11} \left(6x^2 \; - \; 30x \; + \; 400\right) \, dx =& (2x^3 - 15x^2 + 400x + c)|_0^{11} \\ =& 2(11)^3 - 15(11)^2 + 400(11) \\ =& 1287\end{aligned}$$

Problem 8. Do the following problems from the book.

a. Section 9.1
   1) 1a
   2) 1b
   3) 1c
   4) 3a
   5) 3b
   6) 5a
   7) 5b
   8) 5c

b. Section 9.2
   1) 5a
   2) 5b
   3) 5c
   4) 5d
   5) 5e
   6) 7

c. Section 9.3
   1) 1a
   2) 1b
   3) 1c
   4) 1d
   5) 1e