ECONOMICS 207
SPRING 2007
PROBLEM SET 7—KEY

Problem 1. Find the second derivative of each of the following functions with respect to x

a. $f(x) = -300 + 100x + 10x^2 - x^3$

$$\begin{aligned} f'(x) =& 100 + 20x - 3x^2 \\ f''(x) =& 20 - 6x \end{aligned}$$

b. $f(x) = 30x^{2/3}z^{1/4} - 10x - 10z$

$$\begin{aligned} f'(x) =& 20x^{-1/3}z^{1/4} - 10 \\ f''(x) =& -\frac{20}{3}x^{-4/3}z^{1/4} \end{aligned}$$

c. $f(x) = (4x^3 + 2x^2 + 3x)^3$

$$\begin{aligned} f'(x) =& 3(4x^3 + 2x^2 + 3x)^2(12x^2 + 4x + 3) \\ f''(x) =& 3[(4x^3 + 2x^2 + 3x)^2]'(12x^2 + 4x + 3) + 3(4x^3 + 2x^2 + 3x)^2(12x^2 + 4x + 3)' \\ =& 6(4x^3 + 2x^2 + 3x)(12x^2 + 4x + 3) + 3(4x^3 + 2x^2 + 3x)^2(24x + 4) \end{aligned}$$

d. $f(x) = \log[x^3 - 2x^2]$

$$\begin{aligned} f'(x) =& \frac{3x^2 - 4x}{x^3 - 2x^2} \\ f''(x) =& \frac{(3x^2 - 4x)'(x^3 - 2x^2) - (3x^2 - 4x)(x^3 - 2x^2)'}{(x^3 - 2x^2)^2} \\ =& \frac{(6x - 4)(x^3 - 2x^2) - (3x^2 - 4x)(3x^2 - 4x)}{(x^3 - 2x^2)^2} \end{aligned}$$

Date: 18 January 2008.

Problem 2. Find the definite integral of each of the following functions.

a. $\int_1^3 \left(3x^2 \ + \ 3x\right) \, dx$

$$\begin{aligned}\int_1^3 \left(3x^2 \ + \ 3x\right) \, dx =& (x^3 + \frac{3}{2}x^2)|_1^3 \\ =& (3^3 + \frac{3}{2}3^2) - (1^3 + \frac{3}{2}1^2) \\ =& (27 + 27/2) - (1 + 3/2) \\ =& 38\end{aligned}$$

b. $\int_{27}^{216} \left(20x^{-1/3}z^{1/4} \ - \ 10\right) \, dx, \quad z \ = \ 81.$

$$\begin{aligned}\int_{27}^{216} \left(20x^{-1/3}z^{1/4} \ - \ 10\right) \, dx =& (\frac{20}{\frac{2}{3}}x^{1-\frac{1}{3}}z^{1/4} - 10)|_{27}^{216} \\ =& (30x^{2/3}z^{1/4} - 10x)|_{27}^{216} \\ =& (30(216)^{2/3}(81)^{1/4} - 10 \times 216) - (30(27)^{2/3}(81)^{1/4} - 10 \times 27) \\ =& (30(6)^2(3) - 216 \times 10) - (30(3)^2(3) - 270) \\ =& (3240 - 2160) - (1110 - 270) \\ =& 1080 - 840 \\ =& 240\end{aligned}$$

c. $\int_0^{10} \left(3x^2 \ - \ 20x \ + \ 100\right) \, dx$

$$\begin{aligned}\int_0^{10} \left(3x^2 \ - \ 20x \ + \ 100\right) \, dx =& (x^3 - 10x^2 + 100x)_0^{10} \\ =& (10)^3 - 10(10)^2 + 1000 - 0 \\ =& 1000 - 1000 + 1000 \\ =& 1000\end{aligned}$$

d. $\int_0^{10} (250 \ - \ 5x) \, dx$

$$\begin{aligned}\int_0^{10} (250 \ - \ 5x) \, dx =& (250x - \frac{5}{2}x^2)|_0^{10} \\ =& 250 \times 10 - \frac{5}{2}(10)^2 - 0 \\ =& 2500 - \frac{5}{2}100 \\ =& 2500 - 250 \\ =& 2250\end{aligned}$$

Problem 3. Solve the following systems of equations.

$$20x_1^{-1/3}x_2^{1/4} \; - \; 10 \; = 0$$
$$\frac{15}{2}x_1^{2/3}x_2^{-3/4} \; - \; 10 \; = 0$$

First, we move the second terms of both equations to the right-side. And we get

$$20x_1^{-1/3}x_2^{1/4} \; = 10$$
$$\frac{15}{2}x_1^{2/3}x_2^{-3/4} \; = 10$$

Then, we divide two equations and we have

$$\frac{20x_1^{-1/3}x_2^{1/4}}{\frac{15}{2}x_1^{2/3}x_2^{-3/4}} \; = \frac{10}{10}$$
$$\rightarrow \frac{20x_2}{\frac{15}{2}x_1} =1$$
$$\rightarrow \frac{x_2}{x_1} =\frac{\frac{15}{2}}{20}$$
$$\rightarrow \frac{x_2}{x_1} =\frac{3}{8}$$
$$\rightarrow x_2 =\frac{3}{8}x_1$$

We subsitute that into the first equation and we have

$$20x_1^{-1/3}x_2^{1/4} =10$$
$$\rightarrow 20x_1^{-1/3}(\frac{3}{8}x_1)^{1/4} =10$$
$$\rightarrow 20x_1^{-1/3}x_1^{1/4}(3/8)^{1/4} =10$$
$$\rightarrow 2(3/8)^{1/4}x_1^{-1/3+1/4} =1$$
$$\rightarrow 6^{1/4}x_1^{-1/12} =1$$
$$\rightarrow 6^{1/4} =x_1^{1/12}$$
$$\rightarrow (6^{1/4})^{12} =(x_1^{1/12})^{12}$$
$$\rightarrow 6^3 =x_1$$
$$\rightarrow x_1 =216$$

Problem 4.

a. Find the profit maximizing level of output for the following firm. Demonstrate that the level you choose maximizes profit.

$$price \ = \ p \ = \$200$$

$$cost \ = \ c(y) \ = 300 \ + \ 100y \ - \ 10y^2 \ + \ y^3$$

The profit is

$$\begin{aligned}\pi =& py - c(y)\\ =& 200y - 300 - 100y + 10y^2 - y^3\\ =& -300 + 100y + 10y^2 - y^3\end{aligned}$$

We derive the first and second derivatives,

$$\begin{aligned}\pi' =& 100 + 20y - 3y^2\\ \pi'' =& 20 - 6y\end{aligned}$$

To find the optimal output, we set the first derivative to 0, that is,

$$\begin{aligned}\pi' =& 0\\ 100 + 20y - 3y^2 =& 0\\ \rightarrow 3y^2 - 20y - 100 =& 0\\ \rightarrow y = & \frac{20 \pm \sqrt{400 + 3 \cdot 4 \cdot 100}}{6}\\ \rightarrow y = & \frac{20 \pm \sqrt{1600}}{6}\\ \rightarrow y = & 10, -10/3\end{aligned}$$

Check the second derivative,

$$\begin{aligned}\pi''(10) =& 20 - 60\\ =& -40 < 0\end{aligned}$$

So, 10 is the optimal output.

b. What is revenue minus variable cost for this firm when price is $200?

$$\begin{aligned}
\text{Revenue} =& py \\
=& 2000 \\
\text{Variable Cost} =& 100y - 10y^2 + y^3 \\
=& 1000 \\
\text{Revenue-Variable Cost} =& 2000 - 1000 \\
=& 1000
\end{aligned}$$

c. Find producer surplus for this firm assuming you are only given the following marginal cost function: $MC(y) = 100 - 20y + 3y^2$ and a price of $200.

$$\begin{aligned}
\int_0^{10} (p - MC(y))dy =& \int_0^{10} (200 - 100 + 20y - 3y^2)dy \\
=& \int_0^{10} (100 + 20y - 3y^2)dy \\
=& (100y + 10y^2 - y^3)|_0^{10} \\
=& 1000 + 1000 - 1000 \\
=& 1000
\end{aligned}$$

Problem 5.

a. Find the profit maximizing level of output for the following firm. Demonstrate that the level you choose maximizes profit.

$$price \;=\; p \;=\$374$$
$$cost \;=\; c(y) \;= 500 \;+\; 400y \;-\; 40y^2 \;+\; 2y^3$$

The profit is

$$\begin{aligned}\pi =& py - c(y)\\ =& 374y - 500 - 400y + 40y^2 - 2y^3\\ =& -500 - 26y + 40y^2 - 2y^3\end{aligned}$$

The first and second derivatives are

$$\begin{aligned}\pi' =& -26 + 80y - 6y^2\\ \pi'' =& 80 - 12y\end{aligned}$$

First, set the first derivative to 0, and we have

$$\begin{aligned}\pi' =& 0\\ -26 + 80y - 6y^2 =& 0\\ \rightarrow 6y^2 - 80y + 26 =& 0\\ \rightarrow y =& \frac{80 \pm \sqrt{6400 - 6 \cdot 4 \cdot 26}}{12}\\ =& \frac{80 \pm \sqrt{5776}}{12}\\ =& \frac{20 \pm \sqrt{361}}{3}\end{aligned}$$

Check the second derivative

$$\begin{aligned}\pi''(\frac{20 + \sqrt{361}}{3}) =& 80 - 4(\frac{20 + \sqrt{361}}{3})\\ =& -4\sqrt{361} < 0\\ \pi''(\frac{20 - \sqrt{361}}{3}) =& 80 - 4(\frac{20 - \sqrt{361}}{3})\\ =& 4\sqrt{361} > 0\end{aligned}$$

So, $\frac{20+\sqrt{361}}{3}$ is the maximum output.

b. What is revenue minus variable cost for this firm when price is $374?

$$\text{Revenue-Variable Cost} = 374 \cdot \frac{20-\sqrt{361}}{3} - (400y \;-\; 40y^2 \;+\; 2y^3)|_{\frac{20-\sqrt{361}}{3}}$$

c. Find producer surplus for this firm assuming you are only given the following marginal cost function: $MC(y) = 400 - 80y + 6y^2$ and a price of $374.

$$\begin{aligned}\int_0^{\frac{20-\sqrt{361}}{3}} (p - MC(y))dy &= \int_0^{\frac{20-\sqrt{361}}{3}} (374 - 400 + 80y - 6y^2)dy \\ &= \int_0^{\frac{20-\sqrt{361}}{3}} (-26 + 80y - 6y^2)dy \\ &= \int_0^{\frac{20-\sqrt{361}}{3}} (-26y + 40y^2 - 2y^3)dy\end{aligned}$$

Problem 6. In the following problem you are given a production function for a firm where y is the level of output and x is the level of the variable input. You are given the price (p) of the output and the price (w) of the single variable input. Write down an equation that represents profit for the firm. Then maximize this function by taking its derivative with respect to the variable input x and set equal to zero. What is the optimal level of x? Show why this x is the one that maximizes profit.

$$\text{output price} = p = 3$$

$$\text{input price} = w = 792$$

$$y = \text{output} = f(x) = 30x + 40x^2 - 2x^3$$

The profit is

$$\begin{aligned}\pi =& py - wx\\ =& 3(30x + 40x^2 - 2x^3) - 792x\\ =& 90x + 120x^2 - 6x^3 - 792x\\ =& 120x^2 - 6x^3 - 702x\end{aligned}$$

The first and second derivatives are

$$\begin{aligned}\pi' =& 240x - 18x^2 - 702\\ \pi'' =& 240 - 36x\end{aligned}$$

Setting the first derivative to 0, we get

$$\begin{aligned}\pi' =& 0\\ 240x - 18x^2 - 702 =& 0\\ \rightarrow -117 + 40x - 3x^2 =& 0\\ \rightarrow x = \frac{40 \pm \sqrt{1600 - 12 \cdot 117}}{6}&\\ \rightarrow x = \frac{40 \pm \sqrt{196}}{6}&\\ \rightarrow x = 9, 13/3&\end{aligned}$$

Check the second derivative,

$$\begin{aligned}\pi''(9) =& 240 - 36 \cdot 9\\ =& 240 - 324\\ =& -84\\ \pi''(13/3) =& 240 - 36 \cdot \frac{13}{3}\\ =& 240 - 12 \cdot 13\\ =& 240 - 156\\ =& 84\end{aligned}$$

So, 9 is a maximum point.

Problem 7. Solve the following system of equations for x_1, x_2, and x_3.

$$\{x_1 \ = \ 1, \ x_2 \ = \ 3, \ x_3 \ = \ 1\}$$

$$x_1 \ - \ 2x_2 \ + \ 3x_3 \ = \ -2$$

$$4x_1 \ - \ 9x_2 \ + \ 12x_3 \ = \ -11$$

$$-2x_1 \ + \ 3x_2 \ - \ 5x_3 \ = 2$$

We first multiply the first equation by -4 and add it to the second equation, that is,

$$\begin{aligned} -4x_1 + 8x_2 - 12x_3 + 4x_1 - 9x_2 + 12x_3 =& 8 - 11 \\ \rightarrow -x_2 =& -3 \\ \rightarrow x_2 =& 3 \end{aligned}$$

Next, multiply the first equation by 2 and add it to the third equation, that is,

$$\begin{aligned} 2x_1 - 4x_2 + 6x_3 + (-2x_1) + 3x_2 - 5x_3 =& -4 + 2 \\ \rightarrow -x_2 + x_3 =& -2 \\ \rightarrow -3 + x_3 =& -2 \\ \rightarrow x_3 =& 1 \end{aligned}$$

Lastly, we substitute x_2 and x_3 to the first equation, and we solve that $x_1 = 1$.

Problem 8. For each of the following problems, find the critical points. For each critical point state whether the function is at a relative maximum, relative minimum, or otherwise.
a. $y = x^2$

$$\begin{aligned} y' =& 2x \\ y'' =& 2 > 0 \\ y' =& 0 \\ \rightarrow 2x =& 0 \\ \rightarrow x =& 0 \end{aligned}$$

Since $y'' > 0$, 0 is a minimum point.

b. $y = 9x^3 - 27x^2$

$$\begin{aligned} y' =& 27x^2 - 54x \\ y'' =& 54x - 54 \\ y' =& 0 \\ \rightarrow 27x^2 - 54x =& 0 \\ \rightarrow x^2 - 2x =& 0 \\ \rightarrow x =& 0, 2 \\ y''(0) =& -54 < 0 \\ y''(2) =& 54 > 0 \end{aligned}$$

So, 0 is a maximum point and 2 is a minimum point.

c. $f(x) = -960x + 400x^2 - 20x^3$

$$\begin{aligned}
f'(x) &= -960 + 800x - 60x^2 \\
f''(x) &= 800 - 10x \\
f'(x) &= 0 \\
\rightarrow -960 + 800x - 60x^2 &= 0 \\
\rightarrow 6x^2 - 80x + 96 &= 0 \\
\rightarrow x &= \frac{80 \pm \sqrt{6400 - 6 \cdot 4 \cdot 96}}{12} \\
\rightarrow &= \frac{80 \pm \sqrt{4096}}{12} \\
\rightarrow &= \frac{80 \pm 64}{12} \\
\rightarrow &= 12, \frac{4}{3} \\
f''(12) &= 800 - 120 \cdot 12 \\
&= 800 - 1440 \\
&= -640 < 0 \\
f''(4/3) &= 800 - 120 \cdot \frac{4}{3} \\
&= 800 - 160 \\
&= 640
\end{aligned}$$

So, 12 is a maximum point and 4/3 is the minimum point.

d. $f(x) = -20x^3 + 1500x$

$$\begin{aligned}
f'(x) &= -60x^2 + 1500 \\
f''(x) &= -120x \\
f'(x) &= 0 \\
-60x^2 + 1500 &= 0 \\
6x^2 &= 150 \\
x^2 &= 25 \\
x &= \pm 5 \\
f''(5) &= -120 \cdot 5 \\
&= -600 < 0 \\
f''(-5) &= 600 > 0
\end{aligned}$$

So, 5 is a maximum point and -5 is a minimum point.

e. $f(x) = -3x^5 + 5x^3$

$$\begin{aligned}
f'(x) &= -15x^4 + 15x^2 = 0 \\
\Rightarrow -15x(x^3 - x) &= 0 \\
\Rightarrow -15x^2(x^2 - 1) &= 0 \\
\Rightarrow x &= 0 \text{ is a double root} \\
\Rightarrow x &= 1 \text{ is a root and } x = -1 \text{ is a root}
\end{aligned}$$

$$\begin{aligned}
f''(x) &= -60x^3 + 30x^2 \\
f''(1) &= -60(1) + 30(1) = -30 \Rightarrow x = 1 \text{ is a max} \\
f''(-1) &= -60(-1) + 30(-1) = 30 \Rightarrow x = -1 \text{ is a min}
\end{aligned}$$

$$\begin{aligned}
f'''(x) &= -180x^2 + 60x \\
f^4(x) &= -360x + 60 \\
f^5(x) &= -360
\end{aligned}$$

Given that the second, third, and fourth derivatives of f(·) evaluated at x = 0 give a value of zero, and f^5(x)= -360, the test fails for x = 0.

f. $f(x) = x^2 + \frac{1}{x^2}$

$$\begin{aligned}
f'(x) &= 2x - 2x^{-3} = 0 \\
\Rightarrow 2x^{-3}(x^4 - 1) &= 0, \qquad x \neq 0. \\
\Rightarrow x^4 &= 1 \\
\Rightarrow x &= 1, x = -1, x = i, x = -i \text{ are roots of the equation} \\
f''(x) &= 2 + 6x^{-4} \\
f''(1) &= 8 \Rightarrow x = 1 \text{ is a min} \\
f''(-1) &= 8 \Rightarrow x = -1 \text{ is a min}
\end{aligned}$$

Problem 9. Do the following problems from the book.

a. Section 9.4
   1) 3 (Example 3 is useful to study)
   2) 5

b. Section 8.5
   1) 3
   2) 5a

c. Section 8.6
   Problem 1

d. Section 8.7
   You will need to use Theorem 8.7.1 for these problems.
   1) 1a
   2) 1b
   3) 3a
   4) 3b
   5) 3c