ECONOMICS 207
SPRING 2007
PROBLEM SET 8—KEY

Consider the following matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 4 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}, \quad F = \begin{bmatrix} -2 & 3 \\ -3 & 1 \\ 4 & 2 \end{bmatrix}$$

$$a = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad c = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Problem 1. Compute the following

a. A + B

Since A and B have different dimensions, this operation is not performable.

b. D'

$$\text{D'} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}$$

c. AB

$$\text{AB} = \begin{pmatrix} 1*2+2*3 & 1*1+2*4 & 1*6+2*5 \\ 2*2+3*3 & 2*1+3*4 & 2*6+3*5 \end{pmatrix}$$
$$= \begin{pmatrix} 8 & 9 & 16 \\ 13 & 14 & 27 \end{pmatrix}$$

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d. AD

$$\begin{aligned} \text{AD} &= \begin{pmatrix} 1*(-3)+2*2 & 1*2+2*(-1) \\ 2*(-3)+3*2 & 2*2+3*(-1) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

e. ADB

$$\begin{aligned} \text{ADB} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 6 \\ 3 & 4 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 & 6 \\ 3 & 4 & 5 \end{pmatrix} \end{aligned}$$

f. FA

$$\text{FA} = \begin{pmatrix} 4 & 5 \\ -1 & 3 \\ 8 & 14 \end{pmatrix}$$

g. FAD

$$\begin{aligned} \text{FAD} &= \begin{bmatrix} 4 & 5 \\ -1 & -3 \\ 8 & 14 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 3 \\ -3 & 1 \\ 4 & 2 \end{bmatrix} \end{aligned}$$

h. Ca

The dimension of C is $2*2$ and a is $3*1$. They are not compatible, so we cannot perform the multiplication.

i. DACa

The dimension of this multiplication is $(2 \times 2) * (2 \times 2) * (2 \times 2) * (3 \times 1)$ and it is clear that they are not compatible.

j. CF

The dimension of these two matrix are not compatible.

k. b'F

$$
\begin{aligned}
b'F &= \begin{bmatrix} 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ -3 & 1 \\ 4 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 6 & 6 \end{bmatrix}
\end{aligned}
$$

l. c'C

$$
c'C = \begin{bmatrix} 7 & 3 \end{bmatrix}
$$

Problem 2. For each of the following problems, find the critical points. For each critical point state whether the function is at a relative maximum, relative minimum, or otherwise. Also find the points of inflection for each function.

a. $f(x) = 180x^{1/3} - 15x$

The first and second derivatives are:

$$f'(x) = 60x^{-2/3} - 15,$$
$$f''(x) = -40x^{-5/3}.$$

To solve the critical points, we set the first derivative to 0, that is,

$$f'(x) = 0$$
$$60x^{-2/3} - 15 = 0$$
$$4x^{-2/3} - 1 = 0$$
$$4 = x^{2/3}$$
$$(x^{2/3})^{3/2} = 4^{3/2}$$
$$x = 8$$

Next, we consider the second derivative:

$$f''(8) = -40(8^{-5/3}),$$
$$= -5/4 < 0.$$

So 8 is a maximum point.

To find if there are inflection points, we set the second derivative to 0, that is,

$$f''(x) = 0.$$

It is clear to see that there does not have a solution, so there is no other inflection point.

b. $y = 42x^2 - 2x^3 + 600$

$$f'(x) = 84x - 6x^2$$
$$f''(x) = 84 - 12x$$
$$f'(x) = 0$$
$$84x - 6x^2 = 0$$
$$14x - x^2 = 0$$
$$x(14 - x) = 0$$
$$x = 0, 14$$
$$f''(0) = 84 > 0$$
$$f''(14) = -84 < 0$$

Hence, 0 is a minimum point and 14 is a maximum point.

$$\begin{aligned} f''(0) =& 0 \\ 84 - 12x =& 0 \\ 21 - 3x =& 0 \\ x =& 7 \end{aligned}$$

So, 7 is an inflection point.

c. $f(x) \;=\; 84x - 27x^2 + 2x^3$

Here is the graph.

Find the first and second derivations of the function of the function.

$$\begin{aligned} f'(x) &= 6x^2 - 54x + 84 \\ f''(x) &= 12x - 54 \end{aligned}$$

Set the first derivative equal to zero and solve for x.

$$\begin{aligned} f'(x) &= 6x^2 - 54x + 84 \;=\; 0 \\ \Rightarrow (6x - 12)(x - 7) &= 0 \\ \Rightarrow x &= 2 \text{ and } x = 7 \end{aligned}$$

To check which solutions are minimums and which are maximums we consider the second derivatives.

When x = 2 we obtain

$$\begin{aligned} f''(2) &= 12(2) - 54 \\ &= 24 - 54 = -30 \end{aligned}$$

So x = 2 is a maximum point.

When x = 7 we obtain

$$\begin{aligned} f''(2) &= 12(7) - 54 \\ &= 84 - 54 = 30 \end{aligned}$$

So x = 7 is a minimum point.

To find points of inflection we set the second derivative equal to zero.

$$\begin{aligned} f''(x) &= 12x - 54 \;=\; 0 \\ \Rightarrow x &= \frac{54}{12} \;=\; \frac{9}{2} \end{aligned}$$

So $\frac{9}{2}$ is an inflection point.

d. $f(x) = 6x^4 - 4x^3$

The first and second derivatives are

$$f'(x) = 24x^3 - 12x^2,$$
$$f''(x) = 72x^2 - 24x.$$

Setting the first derivative to 0, we have

$$f'(x) = 0$$
$$\Rightarrow 24x^3 - 12x^2 = 0,$$
$$\Rightarrow 2x^3 - x^2 = 0,$$
$$\Rightarrow x^2(2x - 1) = 0,$$
$$\Rightarrow x = 0, 1/2.$$

Next, check the second derivative:

$$f''(0) = 0,$$
$$f''(1/2) = 6 > 0.$$

Since $f''(0) = 0$, so 0 is undetermined. However, 1/2 is a minimum point.

Finally,

$$f''(0) = 0,$$
$$\Rightarrow 72x^2 - 24x = 0,$$
$$\Rightarrow x(72x - 24) = 0,$$
$$\Rightarrow x = 0, 1/3.$$

Since $f'''(0) = 148x - 24 = -24 < 0$, so 0 and 1/3 are inflection points.

e. $\frac{x^4+1}{x^2+1}$, $(x \neq 1, x \neq -1)$, Substitute $z = x^2$ in the first order conditions and then solve for the roots. One will be zero and there will be four others, two are complex numbers. You need not find inflection points for the function.

Here is the graph.

Find the first derivative of the function.

$$\begin{aligned} f'(x) &= \frac{\left(x^2+1\right)\left(4x^3\right) - \left(x^4+1\right)(2x)}{\left(x^2+1\right)^2} \\ &= \frac{4x^5+4x^3-2x^5-2x}{\left(x^2+1\right)^2} \\ &= \frac{2x^5+4x^3-2x}{\left(x^2+1\right)^2} \\ &= \frac{2x(x^4+2x^2-1)}{\left(x^2+1\right)^2} \end{aligned}$$

Set this equal to zero.

$$\begin{aligned} f'(x) &= \frac{2x(x^4+2x^2-1)}{\left(x^2+1\right)^2} = 0 \\ &\Rightarrow x^4+2x^2-1 = 0 \end{aligned}$$

Now make the substitution $z = x^2$ as suggested.

$$\begin{aligned} x^4+2x^2-1 &= 0 \\ \Rightarrow z^2+2z-1 &= 0 \end{aligned}$$

Use the quadratic formula to solve for z.

$$\begin{aligned} z &= \frac{-2 \pm \sqrt{4-(4)(1)(1)}}{2} \\ &= \frac{-2 \pm \sqrt{8}}{2} \\ &= \frac{-2 \pm 2\sqrt{2}}{2} \\ &= -1 \pm \sqrt{2} \end{aligned}$$

Now note that $z = x^2$ so that

$$\begin{aligned} x^2 &= -1 \pm \sqrt{2} \\ \Rightarrow x &= \pm\sqrt{-1 \pm \sqrt{2}} \end{aligned}$$

This gives four roots. They are

i.

$$\begin{aligned} x &= \sqrt{-1+\sqrt{2}} \\ &= \sqrt{-1 + 1.41421} \\ &= \sqrt{0.41421} = 0.643594 \end{aligned}$$

ii.

$$\begin{aligned} x &= \sqrt{-1-\sqrt{2}} \\ &= \sqrt{-1\ -\ 1.41421} \\ &= \sqrt{-2.41421} \\ &= \sqrt{(-1)(2.41421)} \\ &= i\sqrt{2.41421} \\ &= i(1.55377) \end{aligned}$$

iii.

$$\begin{aligned} x &= -\sqrt{-1+\sqrt{2}} \\ &= -\sqrt{-1\ +\ 1.41421} \\ &= -\sqrt{0.41421}\ =\ -0.643594 \end{aligned}$$

iv.

$$\begin{aligned} x &= -\sqrt{-1-\sqrt{2}} \\ &= -\sqrt{-1\ -\ 1.41421} \\ &= -\sqrt{-2.41421} \\ &= -\sqrt{(-1)(2.41421)} \\ &= -i\sqrt{2.41421} \\ &= -i(1.55377) \end{aligned}$$

The second derivative of the function is

$$\begin{aligned} f'(x) &= \frac{2x^5+4x^3-2x}{\left(x^2\ +\ 1\right)^2} \\ f''(x) &= \frac{\left(x^2\ +\ 1\right)^2(10x^4+12x^2-2)-\left(2x^5+4x^3-2x\right)(2)(x^2+1)(2x)}{\left(x^2\ +\ 1\right)^4} \\ &= \frac{\left(x^2\ +\ 1\right)(10x^4+12x^2-2)-\left(2x^5+4x^3-2x\right)(4x)}{\left(x^2\ +\ 1\right)^3} \\ &= \frac{10x^6+12x^4-2x^2+10x^4+12x^2-2-8x^6-16x^4+8x^2}{\left(x^2\ +\ 1\right)^3} \\ &= \frac{2x^6+6x^4+18x^2-2}{\left(x^2\ +\ 1\right)^3} \\ &= \frac{2\left(x^6+3x^4+9x^2-1\right)}{\left(x^2\ +\ 1\right)^3} \end{aligned}$$

AS far as checking for a positive and negative sign we can ignore the 2 and the denominator because the square of any number is positive. So we need to evaluate the sign of $x^6+3x^4+9x^2-1$ for zero and each of the two real roots. When x = 0, this term is equal to -2, so x = 0 is a maximum point.

Now note that

$$
\begin{aligned}
\left(\sqrt{-1+\sqrt{2}}\right)^6 &= \left(-1+\sqrt{2}\right)^3 \\
&= \left(-1+\sqrt{2}\right)\left(-1+\sqrt{2}\right)\left(-1+\sqrt{2}\right) \\
&= \left(1-2\sqrt{2}+2\right)\left(-1+\sqrt{2}\right) \\
&= \left(3-2\sqrt{2}\right)\left(-1+\sqrt{2}\right) \\
&= -3+2\sqrt{2}+3\sqrt{2}-4 \\
&= -7+5\sqrt{2} \\
&= 0.0710678
\end{aligned}
$$

Also note that

$$
\begin{aligned}
\left(\sqrt{-1+\sqrt{2}}\right)^4 &= \left(-1+\sqrt{2}\right)^2 \\
&= \left(-1+\sqrt{2}\right)\left(-1+\sqrt{2}\right) \\
&= 3-2\sqrt{2} \\
&= 0.171573
\end{aligned}
$$

And finally note that

$$
\begin{aligned}
\left(\sqrt{-1+\sqrt{2}}\right)^2 &= -1+\sqrt{2} \\
&= 0.414214
\end{aligned}
$$

Now evaluate $x^6+3x^4+9x^2-1$ at $\sqrt{-1+\sqrt{2}}$ to obtain

$$
\begin{aligned}
x^6+3x^4+9x^2-1 &= \left(\sqrt{-1+\sqrt{2}}\right)^6 + 3\left(\sqrt{-1+\sqrt{2}}\right)^4 + 9\left(\sqrt{-1+\sqrt{2}}\right)^2 - 1 \\
&= -7+5\sqrt{2}+3\left(3-2\sqrt{2}\right)+9\left(-1+\sqrt{2}\right)-1 \\
&= -7+5\sqrt{2}+9-6\sqrt{2}-9+9\sqrt{2}-1 \\
&= -8+8\sqrt{2} \\
&= 3.31371
\end{aligned}
$$

So $\sqrt{-1+\sqrt{2}}$ is a minimum point. Similarly $-\sqrt{-1+\sqrt{2}}$ is a minimum point.

f. $f(x) = \frac{x^2-2x+1}{x+1}, \quad x \neq -1$

Here is the graph.

Find find the first derivative of the function.

$$\begin{aligned} f'(x) &= \frac{(x + 1)(2x-2) - \left(x^2 - 2x + 1\right)(1)}{(x + 1)^2} \\ &= \frac{2x^2 - 2 - x^2 + 2x - 1}{(x + 1)^2} \\ &= \frac{x^2 + 2x - 3}{(x + 1)^2} \end{aligned}$$

Set this equal to zero.

$$\begin{aligned} f'(x) = \frac{x^2 + 2x - 3}{(x + 1)^2} &= 0 \\ \Rightarrow x^2 + 2x - 3 &= 0 \\ \Rightarrow (x+3)(x-1) &= 0 \\ &\Rightarrow x = -3, \text{ and } x = 1 \end{aligned}$$

The second derivative of the function is

$$\begin{aligned} f'(x) &= \frac{x^2 + 2x - 3}{(x + 1)^2} \\ f''(x) &= \frac{(x + 1)^2 (2x+2) - \left(x^2 + 2x - 3\right)(2)(x+1)}{(x^2 + 1)^4} \\ &= \frac{(x + 1)(2x+2) - \left(x^2 + 2x - 3\right)(2)}{(x + 1)^3} \\ &= \frac{2x^2 + 4x + 2 - 2x^2 - 4x + 6}{(x + 1)^3} \\ &= \frac{8}{(x + 1)^3} \end{aligned}$$

When x = -3 we obtain

$$\begin{aligned} f''(-3) &= \frac{8}{(-3 + 1)^3} \\ &= \frac{8}{(-2)^3} \\ &= \frac{8}{-8} = -1 \end{aligned}$$

So x = -3 is a maximum point.

When x = 1 we obtain

$$\begin{aligned} f''(1) &= \frac{8}{(1 + 1)^3} \\ &= \frac{8}{(2)^3} \\ &= \frac{8}{8} = \end{aligned}$$

So x = 1 is a minimum point.

To find points of inflection we set the second derivative equal to zero.

$$f''(x) = \frac{8}{(x + 1)^3} = 0$$

If x $\neq$ -1, we can multiple both sides of the equation by $(x + 1)^3$ obtaining the nonsensical result that 8 = 0. There are no inflection points.

g. $f(x) = 36x + 3x^2 - 8x^3 + \frac{3}{2}x^4$. (The critical values are x = -1, x = 2, and x = 3).

Here is the graph.

Find the first and second derivations of the function of the function.

$$
\begin{aligned}
f'(x) &= 6x^3 - 24x^2 + 6x + 36 \\
f''(x) &= 18x^2 - 48x + 6
\end{aligned}
$$

Set the first derivative equal to zero.

$$f'(x) \;=\; 6x^3 - 24x^2 + 6x + 36 \;=\; 0$$

If there are integer roots to this equation,we can factor the cubic if we know one of them. We know that x = -1 is a root of this equation so we know that we must have $(x+1)(x+a)(x+b) = 6x^3 - 24x^2 + 6x + 36 = 0$. So if we divide $6x^3 - 24x^2 + 6x + 36$ by (x+1) we can obtain a quadratic represented by (x+a)(x+b). Hopefully we can factor this expression. So let's begin.

If we divide $6x^3 - 24x^2 + 6x + 36$ by (x+1) we first obtain

$$
\begin{array}{r|l}
 & 6x^2 \\
\hline
x+1 & 6x^3 - 24x^2 \;\; + 6x + 36 \\
 & -\,6x^3 \;\; -\,6x^2 \\
\cline{2-2}
\end{array}
$$

$$
\begin{array}{r|l}
 & 6x^2 \\
\hline
x+1 & 6x^3 - 24x^2 \;\; + 6x + 36 \\
 & -\,6x^3 \;\; -\,6x^2 \\
\cline{2-2}
 & \qquad -\,30x^2 \;\; +\,6x
\end{array}
$$

Continuing we obtain

$$
\begin{array}{r|l}
 & 6x^2 - 30x \\
\hline
x+1 & 6x^3 - 24x^2 \;\; + 6x + 36 \\
 & -\,6x^3 \;\; -\,6x^2 \\
\cline{2-2}
 & \qquad -\,30x^2 \;\; +\,6x \\
 & \qquad\;\; 30x^2 + 30x \\
\cline{2-2}
\end{array}
$$

and then

$$
\begin{array}{r}
6x^2 - 30x \\
x+1 \overline{\big) \; 6x^3 - 24x^2 + 6x + 36} \\
\underline{-6x^3 - 6x^2} \\
-30x^2 + 6x \\
\underline{30x^2 + 30x} \\
36x + 36
\end{array}
$$

$$
\begin{array}{r}
6x^2 - 30x + 36 \\
x+1 \overline{\big) \; 6x^3 - 24x^2 + 6x + 36} \\
\underline{-6x^3 - 6x^2} \\
-30x^2 + 6x \\
\underline{30x^2 + 30x} \\
36x + 36
\end{array}
$$

$$
\begin{array}{r}
6x^2 - 30x + 36 \\
x+1 \overline{\big) \; 6x^3 - 24x^2 + 6x + 36} \\
\underline{-6x^3 - 6x^2} \\
-30x^2 + 6x \\
\underline{30x^2 + 30x} \\
36x + 36 \\
\underline{-36x - 36} \\
0
\end{array}
$$

We can now factor the remaining term as follows

$$
\begin{aligned}
6x^2 - 30x + 36 &= 6(x^2 - 5x + 6) \\
&= 6(x-3)(x-2)
\end{aligned}
$$

We then have

$$
\begin{aligned}
(x+1)6(x-3)(x-2) &= 0 \\
\Rightarrow x = -1, x = 3, x = 2
\end{aligned}
$$

We can check each of the roots in the second derivative.

$$\begin{aligned} f''(-1) &= 18(-1)^2 - 48(-1) + 6 \\ &= 18 + 48 + 6 \;=\; 72 \end{aligned}$$

So x = -1 is a minimum point.

$$\begin{aligned} f''(2) &= 18(2)^2 - 48(2) + 6 \\ &= 72 - 96 + 6 \;=\; -18 \end{aligned}$$

So x = 2 is a maximum point.

$$\begin{aligned} f''(3) &= 18(3)^2 - 48(3) + 6 \\ &= (18)(9) - (48)(3) + 6 \\ &= 162 - 144 + 6 \;=\; 24 \end{aligned}$$

So x = 3 is a minimum point.
To find the points of inflection we set the second derivative equal to zero.

$$\begin{aligned} f''(x) &= 18(x)^2 - 48(x) + 6 \;=\; 0 \\ &= 6(3x^2 - 8x + 1) \;=\; 0 \end{aligned}$$

And unfortunately it does not factor so we obtain

$$\begin{aligned} x &= \frac{8 \pm \sqrt{64 - (4)(3)(1)}}{6} \\ &= \frac{8 \pm \sqrt{52}}{6} \\ &= \frac{8 \pm 2\sqrt{13}}{6} \\ &= \frac{4 \pm \sqrt{13}}{3} \end{aligned}$$

Problem 3. Find the definite integral of each of the following functions.

a. $\int_2^4 \left(2x^3 \;-\; 6x^2 \;+\; 2x\right)$ dx.

$$\begin{aligned}\int_2^4 \left(2x^3 \;-\; 6x^2 \;+\; 2x\right) \text{ dx} =&(\frac{1}{2}x^4 - 2x^3 + x^2)|_2^4 \\ =&(\frac{1}{2}4^4 - 24^3 + 4^2) - (\frac{1}{2}2^4 - 22^3 + 2^2) \\ =&16 + 4 \\ =&20\end{aligned}$$

b. $\int_7^{10} \left(300 - 36y + 9y^2\right)$ dy.

$$\begin{aligned}\int_7^{10} \left(300 - 36y + 9y^2\right) \text{ dy} =&(300y - 18y^2 + 3y^3)|_7^{10} \\ =&(300(10) - 18(10)^2 + 3(10)^3) - (300(7) - 18(7)^2 + 3(7)^3) \\ =&4200 - 2247 \\ =&1953\end{aligned}$$

Problem 4. Solve the following systems of equations (The answers are $x_1 = 49$ and $x_2 = 32$.)

$$70x_1^{-1/2}x_2^{2/5} - 40 = 0$$
$$56x_1^{1/2}x_2^{-3/5} - 49 = 0$$

First, we move the second term of both equations to the right-hand side. And that gives us

$$70x_1^{-1/2}x_2^{2/5} = 40$$
$$56x_1^{1/2}x_2^{-3/5} = 49$$

Divding the two equations, we get

$$\frac{70x_1^{-1/2}x_2^{2/5}}{56x_1^{1/2}x_2^{-3/5}} = \frac{40}{49}$$
$$\frac{70x_2}{56x_1} = \frac{40}{49}$$
$$\frac{x_2}{x_1} = \frac{40 * 56}{49 * 7}$$
$$\frac{x_2}{x_1} = \frac{32}{49}$$
$$x_2 = \frac{32}{49}x_1$$

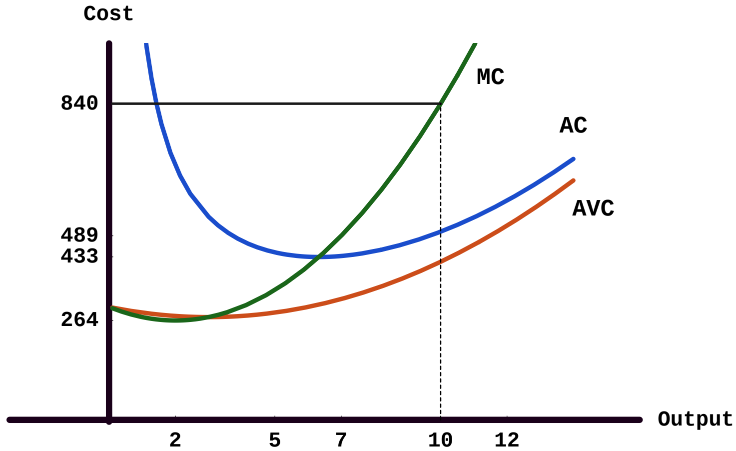
Next, we substitute the last equation to the first equation, and we have

$$70x_1^{-1/2}x_2^{2/5} = 40$$
$$\rightarrow 70x_1^{-1/2}\frac{32}{49}x_1^{\ \ 2/5} = 40$$
$$\rightarrow 7x_1^{-1/2+2/5}\frac{32}{49}^{2/5} = 4$$
$$\rightarrow 7x_1^{-1/10}7^{-4/5}2^2 = 4$$
$$\rightarrow 7^{1-4/5}x_1^{-1/10}4 = 4$$
$$\rightarrow 7^{1/5}x_1^{-1/10} = 1$$
$$\rightarrow 7^{1/5} = x_1^{1/10}$$
$$\rightarrow (x_1^{1/10})^{10} = (7^{1/5})^{10}$$
$$\rightarrow x_1 = 49$$

Substitute x_1 into the equation $x_2 = \frac{32}{49}x_1$, we get $x_2 = 32$.

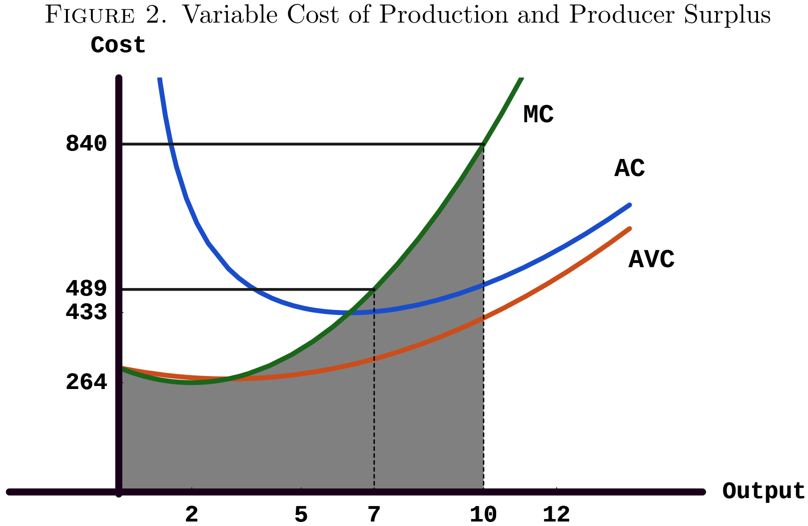
Problem 5. The cost function for a firm is a rule or mapping that tells the total cost of production of any output level produced by the firm. If the variable y represents the output of the firm, then the cost function is given by c(y). Marginal cost represents the change in the cost of production for the firm as output changes and is given by the derivative of the cost function with respect to output,i.e., Marginal Cost(MC) = $\frac{dc(y)}{dy}$. A competitive firm facing a fixed output price maximizes profit at the output level where marginal cost is equal to price. as in the figure 1.

FIGURE 1. Profit Maximization



The area below the cost curve is a measure of variable cost and can be found by integrating the marginal cost curve from 0 to any given output level y. The shaded area in figure 2 represents the variable cost of production for the cost function $c(y) = 800 + 300y - 18y^2 + 3y^3$.

FIGURE 2. Variable Cost of Production and Producer Surplus



Producer surplus is the area below a given price and above the marginal cost curve. Producer surplus is the unshaded area below the horizontal line at 840 in figure 2. Producer surplus can be computed by subtracting the shaded area from total revenue.

a. Find the profit maximizing level of output for the following firm. Demonstrate that the level you choose maximizes profit.

$$price = p = \$840$$
$$cost = c(y) = 800 + 300y - 18y^2 + 3y^3$$

First we find the profit function

$$\begin{aligned}\pi =& py - c(y)\\ =& 840y - 800 - 300y + 18y^2 - 3y^3\\ =& -800 + 540y + 18y^2 - 3y^3\end{aligned}$$

Next, we find the first and second derivatives of π with respect to y.

$$\begin{aligned}\pi' =& 540 + 36y - 9y^2\\ \pi'' =& 36 - 18y\end{aligned}$$

To find the output which maximizes the firm's profit, we set π' to 0,

$$\begin{aligned}\pi' =& 0\\ 540 + 36y - 9y^2 =& 0\\ 60 + 4y - y^2 =& 0\\ y^2 - 4y - 60 =& 0\\ (y - 10)(y + 6) =& 0\\ y = 10, -6\end{aligned}$$

Apparently, 10 is the possible maximum point.
Finally, we check the second derivative, that is,

$$\begin{aligned}\pi''(10) =& 36 - 18 \times 10\\ =& 36 - 180\\ =& -144 < 0.\end{aligned}$$

Hence, 10 is indeed the maximum output.

b. Find the profit maximizing level of output when the price is \$489. Demonstrate that the level you choose maximizes profit.

The profit function now becoms

$$\begin{aligned}\pi =& py - c(y)\\ =& 489y - 800 - 300y + 18y^2 - 3y^3\\ =& -800 + 189y + 18y^2 - 3y^3\end{aligned}$$

And the first and second derivatives of π now change to.

$$\begin{aligned}\pi' =& 189 + 36y - 9y^2\\ \pi'' =& 36 - 18y\end{aligned}$$

Again, we set π' to 0,

$$\begin{aligned}\pi' =& 0\\ 189 + 36y - 9y^2 =& 0\\ 21 + 4y - y^2 =& 0\\ y^2 - 4y - 21 =& 0\\ (y-7)(y+3) =& 0\\ y =& 7, -3\end{aligned}$$

Apparently, 7 is the possible maximum point.
Finally, we check the second derivative, that is,

$$\begin{aligned}\pi''(7) =& 36 - 18 \times 7\\ =& 36 - 126\\ =& -90 < 0.\end{aligned}$$

Hence, 10 is the maximum output when price is 489.

c. What is variable cost for this firm when price is $840?

$$\begin{aligned}\text{Variable cost} =&(300y - 18y^2 + 3y^3)|_{10}\\ =&300 \times 10 - 18 \times 10^2 + 3 \times 10^3\\ =&6000 - 1800\\ =&4200\end{aligned}$$

d. What is producer surplus for this firm when the price is $840?

$$\begin{aligned}MC(y) =&300 - 36y + 9y^2\\ \int_0^{10} [p - Mc(y)]dy =& \int_0^{10} [840 - (300 - 36y + 9y^2)]dy\\ =& \int_0^{10} [540 + 36y - 9y^2)]dy\\ =&(540y + 18y^2 - 3y^3)|_{10}\\ =&4200\end{aligned}$$

e. What is variable cost for this firm when price is $489?

$$\begin{aligned}\text{Variable cost} =&(300y - 18y^2 + 3y^3)|_7\\ =&300 \times 7 - 18 \times 7^2 + 3 \times 7^3\\ =&2247\end{aligned}$$

f. What is producer surplus for this firm when the price is $489?

$$\begin{aligned}MC(y) =&300 - 36y + 9y^2\\ \int_0^{10} [p - Mc(y)]dy =& \int_0^{7} [489 - (300 - 36y + 9y^2)]dy\\ =& \int_0^{7} [189 + 36y - 9y^2)]dy\\ =&(189y + 18y^2 - 3y^3)|_7\\ =&1176\end{aligned}$$

g. How much does variable cost change when output falls from 10 to 7 units?

The variable cost falls by 1953.

h. How much is the firm worse off when price falls from \$840 to \$489?

The firm is worse off by 3024.

i. Cross-hatch the change in producer surplus in Figure 2.

Problem 6. In the following problem you are given a production function for a firm where y is the level of output and x is the level of the variable input. You are given the price (p) of the output and the price (w) of the single variable input. Write down an equation that represents profit for the firm. Then maximize this function by taking its derivative with respect to the variable input x and set equal to zero. What is the optimal level of x? Show why this x is the one one that maximizes profit.

$$\textit{output price} = p = 1$$

$$\textit{input price} = w = 976$$

$$y = \textit{output} = f(x) = 400x + 100x^2 - 4x^3$$

First, we find the profit

$$\begin{aligned} \pi =& py - wx \\ =& 400x + 100x^2 - 4x^3 - 976x \\ =& -576x + 100x^2 - 4x^3 \end{aligned}$$

Then we find the first and second derivatives:

$$\begin{aligned} \pi' =& -576 + 200x - 12x^2 \\ \pi'' =& 200 - 24x \end{aligned}$$

We set the first derivative to 0,

$$\begin{aligned} \pi' =& 0 \\ -576 + 200x - 12x^2 =& 0 \\ -144 + 50x - 3x^2 =& 0 \\ 3x^2 - 50x + 144 =& 0 \\ x =& \frac{50 \pm \sqrt{2500 - 3 \times 4 \times 144}}{6} \\ =& \frac{25 \pm \sqrt{193}}{6} \end{aligned}$$

Check the second derivative,

$$\begin{aligned} \pi''(\frac{25 + \sqrt{193}}{6}) =& 200 - 24 \cdot \frac{25 + sqrt193}{6} \\ =& 200 - 8 \cdot \frac{25 + sqrt193}{6} \\ =& 200 - 200 - 8\sqrt{193} \\ =& -8\sqrt{193} < 0 \\ \pi''(\frac{25 - \sqrt{193}}{6}) =& 200 - 24 \cdot \frac{25 - sqrt193}{6} \\ =& 8\sqrt{193} > 0 \end{aligned}$$

So, $\frac{25+\sqrt{193}}{6}$ is a maximum point and $\frac{25-\sqrt{193}}{6}$ is a minimum point.

Problem 7. Solve the following system of equations for x_1, x_2, and x_3.

$$\{x_1 = 2,\ x_2 = 2,\ x_3 = -1\}$$

$$\begin{aligned} x_1 + 4x_2 + 2x_3 &= 8 \\ -2x_1 - 9x_2 - 4x_3 &= -18 \\ 3x_1 + 6x_2 + 5x_3 &= 13 \end{aligned}$$

First, mlutiply the first equation by 2 and add it to the second one, we get

$$x_2 = 2 \tag{1}$$

Then, multiply the first equation by -3 and add it to the third one and we have

$$-6x_2 - x_3 = -11 \tag{2}$$

Then substitute $x_2 = 2$ into the above equation, we get

$$-12 - x_3 = -11 \tag{3}$$

And that is $x_3 = -1$.
Finally, we put $x_2 = 2$ and $x_3 = -1$ into the first equation, and we solve out x_1.

$$\begin{aligned} x_1 + 8 - 2 &= 8 \\ x_1 &= 2 \end{aligned}$$