ECONOMICS 207
SPRING 2008
EXAM 2
KEY

Arithmetic is where numbers fly like pigeons in and out of your head.
Arithmetic tells you how many you lose or win if you know how many you had before you lost or won.
Arithmetic is seven eleven all good children go to heaven—or five six bundle of sticks.
Arithmetic is numbers you squeeze from your head to your hand to your pencil to your paper till you get the answer.
Arithmetic is where the answer is right and everything is nice and you can look out of the window and see the blue sky—or the answer is wrong and you have to start all over and try again and see how it comes out this time.
If you take a number and double it and double it again and then double it a few more times, the number gets bigger and bigger and goes higher and higher and only arithmetic can tell you what the number is when you decide to quit doubling.
Arithmetic is where you have to multiply—and you carry the multiplication table in your head and hope you won't lose it.
If you have two animal crackers, one good and one bad, and you eat one and a striped zebra with streaks all over him eats the other, how many animal crackers will you have if somebody offers you five six seven and you say No no no and you say Nay nay nay and you say Nix nix nix?
If you ask your mother for one fried egg for breakfast and she gives you two fried eggs and you eat both of them, who is better in arithmetic, you or your mother?
—Carl Sandburg

Notes.

(1) $2^8 = 256$
(2) $2^{11} = 2,048$
(3) $2^{13} = 8,192$
(4) $16 \times 31 = 496$
(5) $496 - 960 = -464$

Date: March 11, 2008.

Problem 1 (15 Points).

Find the derivatives of each of the following functions with respect to x.

a.

$$\begin{aligned} f(x) &= 8x^3 - 5x^2 + 4x + 3x^{-1/3} + 10 \\ \frac{df(x)}{dx} &= 24x^2 - 10x + 4 - x^{-4/3} \end{aligned}$$

b.

$$\begin{aligned} f(x) &= 3x^2 e^{3x^4 - 2x^2} \\ \frac{df(x)}{dx} &= 3x^2 e^{3x^4 - 2x^2}(12x^2 - 4x) \quad + \quad 6xe^{3x^4 - 2x^2} \end{aligned}$$

c.

$$\begin{aligned} f(x) &= \frac{(5x^3 - 2x^2)^2}{2x^3 + 4x} \\ \frac{df(x)}{dx} &= \frac{(2x^3 + 4x)(2)(5x^3 - 2x^2)(15x^2 - 4x) \quad - \quad (5x^3 - 2x^2)^2(6x^2 + 4)}{(2x^3 + 4x)^2} \end{aligned}$$

Problem 2 (15 Points)**.**

Find the second derivatives of each of the following functions with respect to x.

a.

$$
\begin{aligned}
f(x) &= 12x^3 - 5x^2 + 6x \; + \; e^{2x} \\
\frac{df(x)}{dx} &= 36x^2 - 10x + 6 + 2e^{2x} \\
\frac{d^2 f(x)}{dx^2} &= 72x - 10 + 4e^{2x}
\end{aligned}
$$

b.

$$
\begin{aligned}
f(x) &= 256x^{3/8} - 3x \\
\frac{df(x)}{dx} &= 96x^{-5/8} - 3 \\
\frac{d^2 f(x)}{dx^2} &= -\,60x^{-13/8}
\end{aligned}
$$

c.

$$
\begin{aligned}
f(x) &= e^{3x^2+5x} \\
\frac{df(x)}{dx} &= e^{3x^2+5x}(6x+5) \\
\frac{d^2 f(x)}{dx^2} &= e^{3x^2+5x}(6) \qquad + \quad (6x+5)e^{3x^2+5x}(6x+5) \\
&= 6e^{3x^2+5x} \qquad + \quad e^{3x^2+5x}(6x+5)^2
\end{aligned}
$$

Problem 3 (20 Points)**.**

For each of the following problems find the value or values of x where the first derivative of the function is equal to zero. Then decide if the function has a maximum or a minimum or an inflection point at that value of x.

a.

Find the first and second derivatives of f.

$$\begin{aligned} f(x) &= x^3 - 2x^2 - 15x \\ \frac{df(x)}{dx} &= 3x^2 - 4x - 15 \\ \frac{d^2f(x)}{dx^2} &= 6x - 4 \end{aligned}$$

Set the first derivative of f equal to zero and solve for the critical values of x.

$$\begin{aligned} \frac{df(x)}{dx} \;=\; 3x^2 - 4x - 15 &= 0 \\ \Rightarrow (3x+5)(x-3) &= 0 \\ \Rightarrow (3x+5) \;=\; 0 \text{ or } (x-3) &= 0 \\ \Rightarrow x \;=\; \frac{-5}{3} \text{ or } x &= 3 \end{aligned}$$

Now evaluate the second derivative of f(x) at each critical value of x. First for $x = \frac{-5}{3}$

$$\begin{aligned} \frac{d^2\,Profit}{dx^2} &= 6x - 4 \\ \frac{d^2\,Profit}{dx^2}\left(\frac{-5}{3}\right) &= 6\left(\frac{-5}{3}\right) - 4 \\ &= -10 - 4 = -14 \\ \Rightarrow x = \frac{-5}{3} &\text{ is a maximum} \end{aligned}$$

Now for $x = 3$

$$\begin{aligned} \frac{d^2\,Profit}{dx^2} &= 6x - 4 \\ \frac{d^2\,Profit}{dx^2}(3) &= 6\,(3) - 4 \\ &= 18 - 4 = 14 \\ \Rightarrow x = 3 &\text{ is a minimum} \end{aligned}$$

b.

Find the first and second derivatives of f.

$$\begin{aligned} f(x) &= 256x^{3/8} - 3x \\ \frac{df(x)}{dx} &= 96x^{-5/8} - 4 \\ \frac{d^2 f(x)}{dx^2} &= -60x^{-13/8} \end{aligned}$$

Set the first derivative of f equal to zero and solve for the critical values of x.

$$\begin{aligned} \frac{df(x)}{dx} &= 96x^{-5/8} - 4 = 0 \\ \Rightarrow x^{-5/8} &= \frac{4}{96} = \frac{1}{32} \\ \Rightarrow x &= \left(\frac{1}{32}\right)^{-8/5} = (32)^{8/5} \\ &= \left(2^5\right)^{8/5} = 2^8 = 256 \end{aligned}$$

Now evaluate the second derivative of f(x) at the critical value of x.

$$\begin{aligned} \frac{d^2 f(x)}{dx^2} &= -60x^{-13/8} \\ \frac{d^2 f(x)}{dx^2}(256) &= (-60)(256)^{-13/8} = (-60)(2^8)^{-13/8} \\ &= (-60)(2)^{-13} = -(2^2 \times 3 \times 5 \times 2^{-13}) \\ &= -\frac{15}{2^{11}} = -\frac{15}{2048} \end{aligned}$$

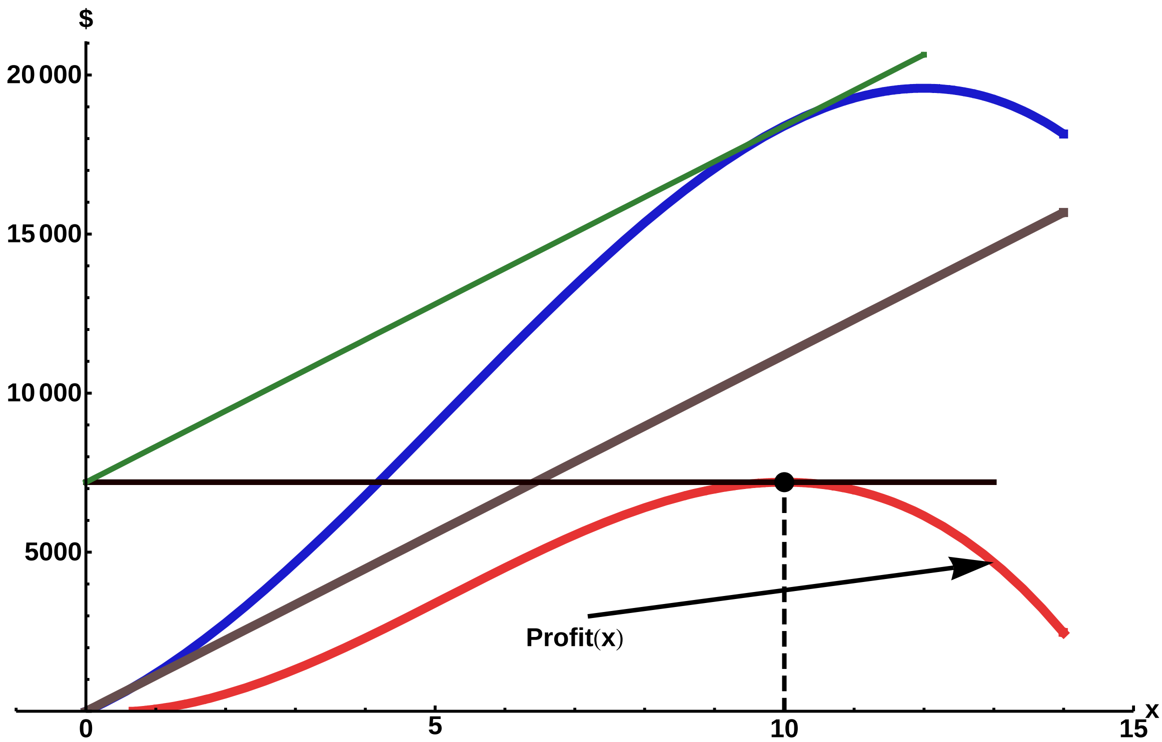
$\Rightarrow x = 256$ is a maximum

Problem 4 (20 Points)**.** In the following problem you are given a production function for a firm where y is the level of output and x is the level of the variable input. You are given the price (p) of the output and the price (w) of the single variable input.

a. Write down an equation that represents profit for the firm.
b. Maximize this function by taking its derivative with respect to the variable input x and setting the derivative equal to zero.
c. Solve the equation in part 4b for the potentially profit maximizing level of x.
d. Determine using the second order conditions which of the roots represents maximum profit.

$$
\begin{aligned}
output\ price &= p = 8 \\
input\ price &= w = 1120 \\
y = output &= f(x) = 120x + 31x^2 - 2x^3
\end{aligned}
$$

Figure 1. Revenue, Cost and Profit



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For this example, profit is given by

$$
\begin{aligned}
Profit &= 8(120x + 31x^2 - 2x^3) - 1120x \\
&= 960x + 248x^2 - 16x^3 - 1120x \\
&= -160x + 248x^2 - 16x^3
\end{aligned}
$$

Extra Space for Problem 4

For this example, the derivative of profit is given by

$$\begin{aligned} Profit &= -16x^3 + 248x^2 - 160x \\ \frac{d\,Profit}{dx} &= -48x^2 + 496x - 160 \\ \frac{d^2\,Profit}{dx^2} &= -96x + 496 \end{aligned}$$

Now set the derivative of profit equal to zero and solve for x as follows.

$$\begin{aligned} \frac{d\,Profit}{dx} = -48x^2 + 496x - 160 &= 0 \\ \Rightarrow -16(3x^2 - 31x + 10) &= 0 \\ \Rightarrow (3x^2 - 31x + 10) &= 0 \\ \Rightarrow (3x - 1)(x - 10) &= 0 \\ \Rightarrow x = \frac{1}{3} \text{ or } x = 10 \end{aligned}$$

Now consider the value of the second derivative at the two roots. First for $x = \frac{1}{3}$.

$$\begin{aligned} \frac{d^2\,Profit}{dx^2} &= -96x + 496 \\ \frac{d^2\,Profit}{dx^2}\left(\frac{1}{3}\right) &= -96\left(\frac{1}{3}\right) + 496 \\ &= -32 + 496 = 464 \quad \Rightarrow x = \frac{1}{3} \text{ is a minimum} \end{aligned}$$

Now for $x = 10$.

$$\begin{aligned} \frac{d^2\,Profit}{dx^2} &= -96x + 496 \\ \frac{d^2\,Profit}{dx^2}(10) &= -96\,(10) + 496 \\ &= -960 + 496 = -464 \quad \Rightarrow x = 10 \text{ is a maximum} \end{aligned}$$

Problem 5 (10 Points)**.**

a. Find the indefinite integral for the following function. Write in the form F(x) + c.

$$f(x) = 96x^{-5/8} - 3$$
$$F(x) = 256x^{3/8} - 3x + c$$

b. Find the indefinite integral for the following function. Write in the form F(x) + c.

$$f(x) = -48x^2 + 496x - 160$$
$$F(x) = -16x^3 + 248x^2 - 160x + c$$

Problem 6 (10 Points).

Solve the following system of equations for x_1, x_2, and x_3.

$$x_1 + 2x_2 - 3x_3 = 2 \quad (6.1)$$

$$-2x_1 - 3x_2 + 8x_3 = 0 \quad (6.2)$$

$$3x_1 + 5x_2 - 10x_3 = 3 \quad (6.3)$$

Choose the first variable and the first equation. Since the coefficient of x is 1, we can proceed directly. We will add (2) times it to the second equation and (-3) times it to the third equation. We will first use (-2) as the multiplier and add the first equation to the second.

$$\begin{array}{rrrcr} 2x_1 & +4x_2 & -6x_3 & = & 4 \\ -2x_1 & -3x_2 & +8x_3 & = & 0 \\ --- & --- & --- & = & -- \\ & +x_2 & +2x_3 & = & 4 \end{array}$$

Now rewrite the system as follows.

$$\begin{array}{rrrcr} x_1 & +2x_2 & -3x_3 & = & 2 \\ & +x_2 & +2x_3 & = & 4 \\ 3x_1 & +5x_2 & -10x_3 & = & 3 \end{array}$$

Now add the (-3) times first equation to the third equation as follows

$$\begin{array}{rrrcr} -3x_1 & -6x_2 & +9x_3 & = & -6 \\ 3x_1 & +5x_2 & -10x_3 & = & 3 \\ --- & --- & --- & = & -- \\ & -x_2 & -x_3 & = & -3 \end{array}$$

Now rewrite the system as follows.

$$\begin{array}{rrrcr} x_1 & +2x_2 & -3x_3 & = & 2 \\ & +x_2 & +2x_3 & = & 4 \\ & -x_2 & -x_3 & = & -3 \end{array}$$

The variable x_1 is now eliminated from all but the first equation. Now choose the second equation and x_2 as the variable. Since its coefficient is 1 we can simply add it to the third equation as follows.

$$\begin{array}{rrrcr} & +x_2 & +2x_3 & = & 4 \\ & -x_2 & -x_3 & = & -3 \\ --- & --- & --- & = & -- \\ & & +x_3 & = & 1 \end{array}$$

The system is now given by

$$\begin{array}{rrrcr} x_1 & +2x_2 & -3x_3 & = & 2 \\ & +x_2 & +2x_3 & = & 4 \\ & & +x_3 & = & 1 \end{array}$$

Now multiply the third equation by -2 and add to the second equation. This will give

$$\begin{array}{rrrcr} & +x_2 & +2x_3 & = & 4 \\ & & -2x_3 & = & -2 \\ --- & --- & --- & = & -- \\ & +x_2 & & = & 2 \end{array}$$

The system is now given by

$$\begin{array}{rrrcr} x_1 & +2x_2 & -3x_3 & = & 2 \\ & +x_2 & & = & 2 \\ & & +x_3 & = & 1 \end{array}$$

Now multiply the third equation by 3 and add to the first equation.

$$\begin{array}{rrrcr} x_1 & +2x_2 & -3x_3 & = & 2 \\ & & +3x_3 & = & 3 \\ --- & --- & --- & = & -- \\ x_1 & +2x_2 & & = & 5 \end{array}$$

The system is now given by

$$\begin{array}{rrrcr} x_1 & +2x_2 & & = & 5 \\ & +x_2 & & = & 2 \\ & & +x_3 & = & 1 \end{array}$$

Now multiply the second equation by -2 and add to the first equation. This will give

$$\begin{array}{rrrcr} x_1 & +2x_2 & & = & 5 \\ & -2x_2 & & = & -4 \\ --- & --- & --- & = & -- \\ x_1 & & & = & 1 \end{array}$$

The system is now given by

$$\begin{array}{rrrcr} x_1 & & & = & 1 \\ & +x_2 & & = & 2 \\ & & +x_3 & = & 1 \end{array}$$

and we are done.

Problem 7 (10 points)**.**
Solve the following systems of equations for x_1 and x_2 using the method of substitution.

$$12x_1^{-2/3}x_2^{1/2} - 8 = 0 \tag{7.1}$$

$$18x_1^{1/3}x_2^{-1/2} - 9 = 0 \tag{7.2}$$

Rearrange the first equation 7.1 to obtain

$$\begin{aligned}
x_1^{-2/3}x_2^{1/2} &= \frac{8}{12} = \frac{2}{3} \\
\Rightarrow x_1^{2/3}x_1^{-2/3}x_2^{1/2} &= \frac{2}{3}x_1^{2/3} \\
\Rightarrow x_2^{1/2} &= \frac{2}{3}x_1^{2/3} \qquad (7.1.a) \\
\Rightarrow x_2 &= \left(\frac{2}{3}\right)^2 \left(x_1^{2/3}\right)^2 \\
&= \left(\frac{2}{3}\right)^2 x_1^{4/3}
\end{aligned}$$

Rearrange the second equation 7.2 slightly to obtain

$$x_1^{1/3}x_2^{-1/2} = \frac{9}{18} = \frac{1}{2} \tag{7.2$'$}$$

Now substitute x_2 from equation 7.1.a into equation 7.2′ to obtain

$$\begin{aligned}
x_1^{1/3}\left(\left(\frac{2}{3}\right)^2 x_1^{4/3}\right)^{-1/2} &= \frac{1}{2} \\
\Rightarrow x_1^{1/3}\left(\frac{2}{3}\right)^{-1} x_1^{-2/3} &= \frac{1}{2} \\
\Rightarrow x_1^{-1/3}\left(\frac{2}{3}\right)^{-1} &= \frac{1}{2} \\
\Rightarrow x_1^{-1/3} &= \frac{1}{2}\left(\frac{2}{3}\right) \qquad (7.2.a) \\
\Rightarrow x_1 &= \left(\frac{1}{2}\left(\frac{2}{3}\right)\right)^{-3} \\
&= 2^3\left(\frac{3}{2}\right)^3 \\
&= (2^3)(3^3)(2^{-1})^3 \\
&= 3^3 = 27
\end{aligned}$$

Now substitute x_1 from equation 7.2.a into equation 7.1.a to obtain

$$\begin{aligned} x_2 &= \left(\frac{2}{3}\right)^2 x_1^{4/3} \\ &= \left(\frac{2}{3}\right)^2 (3^3)^{4/3} \\ &= (2^2)(3^{-1})^2 3^4 \\ &= (2^2)(3^2) = 4 \times 9 = 36 \end{aligned}$$