ECONOMICS 207
EXAM 4
SPRING 2008
KEY

Hints.

(1) Remember rules about dividing by 3 and 9.

(2) $\frac{171}{9} = 19$

(3) $19 \times 18 = 342$

(4) $729 \times 3 = 2187$

(5) $81 \times 50 = 4050$

Problem 1 (25 Points)**.** Consider the following matrix and vector.

$$P = \begin{bmatrix} 1 & 4 \\ -2 & -5 \end{bmatrix}, \quad p = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

a. Use elementary row operations to find the inverse of P and solve the equation Px=p in one set of operations.

$$\begin{aligned} Px &= p \\ \begin{pmatrix} 1 & 4 \\ -2 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned} \tag{1.1}$$

First create the augmented matrix $\tilde{A}$.

$$\tilde{A} = \begin{pmatrix} 1 & 4 & 1 & 0 & 1 \\ -2 & -5 & 0 & 1 & 1 \end{pmatrix}$$

Use row operations to "knock out" a_{21} by turning it into a zero. This would then make the second row have one more leading zero than the first row. We can turn the a_{21} element into a zero by adding 2 times the first row to the second row.

$$\begin{aligned} &2 \times \begin{pmatrix} 1 & 4 & 1 & 0 & 1 \end{pmatrix} \\ &\Rightarrow \begin{pmatrix} 2 & 8 & 2 & 0 & 2 \end{pmatrix} \end{aligned}$$

$$\begin{array}{ccccc} 2 & 8 & 2 & 0 & 2 \\ -2 & -5 & 0 & 1 & 1 \\ -- & -- & -- & -- & -- \\ 0 & 3 & 2 & 1 & 3 \end{array}$$

This will give a new matrix on which to operate. Call it $\tilde{A}_{21}$.

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$$\tilde{A}_{21} = \begin{pmatrix} 1 & 4 & 1 & 0 & 1 \\ 0 & 3 & 2 & 1 & 3 \end{pmatrix} \tag{1.2}$$

Now divide the second row of $\tilde{A}_{21}$ by 3 to obtain

$$\tilde{A}_{22} = \begin{pmatrix} 1 & 4 & 1 & 0 & 1 \\ 0 & 1 & 2/3 & 1/3 & 1 \end{pmatrix} \tag{1.3}$$

Now multiply the second row of $\tilde{A}_{22}$ by -4 and add to the first row.

$$\begin{aligned} -4 \times & \begin{pmatrix} 0 & 1 & 2/3 & 1/3 & 1 \end{pmatrix} \\ \Rightarrow & \begin{pmatrix} 0 & -4 & -8/3 & -4/3 & -4 \end{pmatrix} \end{aligned}$$

$$\begin{array}{ccccc} 0 & -4 & -8/3 & -4/3 & -4 \\ 1 & 4 & 1 & 0 & 1 \\ -- & -- & -- & -- & -- \\ 1 & 0 & -5/3 & -4/3 & -3 \end{array}$$

Now rewrite the matrix as follows.

$$\tilde{A}_{12} = \begin{pmatrix} 1 & 0 & -5/3 & -4/3 & -3 \\ 0 & 1 & 2/3 & 1/3 & 1 \end{pmatrix} \tag{1.4}$$

This implies the following.

$$P^{-1} = \begin{pmatrix} -5/3 & -4/3 \\ 2/3 & 1/3 \end{pmatrix}, \quad x = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \tag{1.5}$$

b. Find the determinant of the matrix P.

$$P = \begin{bmatrix} 1 & 4 \\ -2 & -5 \end{bmatrix}, \quad p = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$|P| = -5 - (-8) = 3$

c. Find the inverse of the matrix P using the cofactor/adjoint method.

The cofactor matrix is given by

$$P_{ij} = \begin{bmatrix} -5 & 2 \\ -4 & 1 \end{bmatrix}$$

The adjoint matrix is given by

$$adj(P) = P^{+} = P'_{ij} = \begin{bmatrix} -5 & -4 \\ 2 & 1 \end{bmatrix}$$

The inverse is given by

$$\frac{adj(P)}{|P|} = \frac{adj(P)}{3} = \frac{\begin{bmatrix} -5 & -4 \\ 2 & 1 \end{bmatrix}}{3} = \begin{bmatrix} \frac{-5}{3} & \frac{-4}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

d. Solve the equation Px=p using the inverse you found in part 1c

$$x = P^{-1}p$$

$$\begin{bmatrix} \frac{-5}{3} & \frac{-4}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-5}{3} + \frac{-4}{3} \\ \frac{2}{3} + \frac{1}{3} \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

e. Solve the equation Px=p using Cramer's rule.

$$P = \begin{bmatrix} 1 & 4 \\ -2 & -5 \end{bmatrix}, \quad p = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

First solve for x_1.

$$x_1 = \frac{\begin{vmatrix} 1 & 4 \\ 1 & -5 \end{vmatrix}}{\begin{vmatrix} 1 & 4 \\ -2 & -5 \end{vmatrix}} = \frac{-9}{3} = -3$$

Now solve for x_2.

$$x_2 = \frac{\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 4 \\ -2 & -5 \end{vmatrix}} = \frac{3}{3} = 1$$

Problem 2 (30 Points).

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -3 & 7 & 3 \\ 4 & -6 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ -8 \\ 7 \end{bmatrix}$$

a. Use elementary row operations to find the inverse of A and solve the equation Ax=b in one set of operations.

First create the augmented matrix $\tilde{A}$.

$$\tilde{A} = \begin{pmatrix} 1 & -2 & -1 & 1 & 0 & 0 & 2 \\ -3 & 7 & 3 & 0 & 1 & 0 & -8 \\ 4 & -6 & -3 & 0 & 0 & 1 & 7 \end{pmatrix}$$

Use row operations to "knock out" a_{21} by turning it into a zero. This would then make the second row have one more leading zero than the first row. We can turn the a_{21} element into a zero by adding 3 times the first row to the second row.

$$3 \times \begin{pmatrix} 1 & -2 & -1 & 1 & 0 & 0 & 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3 & -6 & -3 & 3 & 0 & 0 & 6 \end{pmatrix}$$

3	−6	−3	3	0	0	6
−3	7	3	0	1	0	−8
−−	−−	−−	−−	−−	−−	−−
0	1	0	3	1	0	−2

This will give a new matrix on which to operate. Call it $\tilde{A}_{21}$.

$$\tilde{A}_{21} = \begin{pmatrix} 1 & -2 & -1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 & 1 & 0 & -2 \\ 4 & -6 & -3 & 0 & 0 & 1 & 7 \end{pmatrix} \tag{2.1}$$

Now multiply the first row of $\tilde{A}_{21}$. by -4 and add to the third row. First multiply, then add.

$$-4 \times \begin{pmatrix} 1 & -2 & -1 & 1 & 0 & 0 & 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -4 & 8 & 4 & -4 & 0 & 0 & -8 \end{pmatrix}$$

−4	8	4	−4	0	0	−8
4	−6	−3	0	0	1	7
−−	−−	−−	−−	−−	−−	−−
0	2	1	−4	0	1	−1

This will give a new matrix on which to operate. Call it $\tilde{A}_{31}$.

$$\tilde{A}_{31} = \begin{pmatrix} 1 & -2 & -1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 & 1 & 0 & -2 \\ 0 & 2 & 1 & -4 & 0 & 1 & -1 \end{pmatrix} \tag{2.2}$$

Now multiply the second row of $\tilde{A}_{31}$. by -2 and add to the third row. First multiply, then add.

$$-2 \times \begin{pmatrix} 0 & 1 & 0 & 3 & 1 & 0 & -2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & -2 & 0 & -6 & -2 & 0 & 4 \end{pmatrix}$$

$$\begin{array}{ccccccc} 0 & -2 & 0 & -6 & -2 & 0 & 4 \\ 0 & 2 & 1 & -4 & 0 & 1 & -1 \\ -- & -- & -- & -- & -- & -- & -- \\ 0 & 0 & 1 & -10 & -2 & 1 & 3 \end{array}$$

This will give a new matrix on which to operate. Call it $\tilde{A}_{32}$.

$$\tilde{A}_{32} = \begin{pmatrix} 1 & -2 & -1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 & 1 & 0 & -2 \\ 0 & 0 & 1 & -10 & -2 & 1 & 3 \end{pmatrix} \tag{2.3}$$

We now have a lower triangular matrix and the second and third rows are in reduced row echelon form. We know then at this point that $x_3 = 3$ and $x_2 = -2$. We also know the second and third rows of the inverse.

Next we want to knock out the a_{13} element of $\tilde{A}_{32}$. We do so by adding the third row to the first row.

$$\begin{array}{ccccccc} 0 & 0 & 1 & -10 & -2 & 1 & 3 \\ 1 & -2 & -1 & 1 & 0 & 0 & 2 \\ -- & -- & -- & -- & -- & -- & -- \\ 1 & -2 & 0 & -9 & -2 & 1 & 5 \end{array}$$

This will give a new matrix on which to operate. Call it $\tilde{A}_{13}$.

$$\tilde{A}_{13} = \begin{pmatrix} 1 & -2 & 0 & -9 & -2 & 1 & 5 \\ 0 & 1 & 0 & 3 & 1 & 0 & -2 \\ 0 & 0 & 1 & -10 & -2 & 1 & 3 \end{pmatrix} \tag{2.4}$$

Now knock out a_{12} by multiplying the second row of $\tilde{A}_{13}$ by 2 and adding to the first row. First multiply, then add.

$$\begin{aligned} 2 \times & \begin{pmatrix} 0 & 1 & 0 & 3 & 1 & 0 & -2 \end{pmatrix} \\ \Rightarrow & \begin{pmatrix} 0 & 2 & 0 & 6 & 2 & 0 & -4 \end{pmatrix} \end{aligned}$$

$$\begin{array}{ccccccc} 0 & 2 & 0 & 6 & 2 & 0 & -4 \\ 1 & -2 & 0 & -9 & -2 & 1 & 5 \\ -- & -- & -- & -- & -- & -- & -- \\ 1 & 0 & 0 & -3 & 0 & 1 & 1 \end{array}$$

This will give a new matrix on which to operate. Call it $\tilde{A}_{12}$.

$$\tilde{A}_{12} = \begin{pmatrix} 1 & 0 & 0 & -3 & 0 & 1 & 1 \\ 0 & 1 & 0 & 3 & 1 & 0 & -2 \\ 0 & 0 & 1 & -10 & -2 & 1 & 3 \end{pmatrix} \tag{2.5}$$

This then implies that

$$A^{-1} = \begin{bmatrix} -3 & 0 & 1 \\ 3 & 1 & 0 \\ -10 & -2 & 1 \end{bmatrix}$$

and

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

b. Show that the determinant of the matrix A is equal to one, i.e., $|A| = 1$.

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -3 & 7 & 3 \\ 4 & -6 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ -8 \\ 7 \end{bmatrix}$$

$$|A| = ((1 \times 7 \times -3) + (-2 \times 3 \times 4) + (-1 \times -3 \times -6)) - ((-1 \times 7 \times 4) + (-2 \times -3 \times -3) + (1 \times 3 \times -6))$$

$$=(-21 - 24 - 18) - (-28 - 18 - 18)$$

$$=(-63) - (-64) = 1$$

c. Find the inverse of the matrix A using the cofactor/adjoint method.

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -3 & 7 & 3 \\ 4 & -6 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ -8 \\ 7 \end{bmatrix}$$

We find the cofactors one at a time as follows.

$$\begin{aligned}
c_{11} &= (-1)^{1+1} \begin{vmatrix} 7 & 3 \\ -6 & -3 \end{vmatrix} = (1)\,[(-21) - (-18)] = -3 \\
c_{12} &= (-1)^{1+2} \begin{vmatrix} -3 & 3 \\ 4 & -3 \end{vmatrix} = (-1)\,[(9) - (12)] = 3 \\
c_{13} &= (-1)^{1+3} \begin{vmatrix} -3 & 7 \\ 4 & -6 \end{vmatrix} = (1)\,[(18) - (28)] = -10 \\
c_{21} &= (-1)^{2+1} \begin{vmatrix} -2 & -1 \\ -6 & -3 \end{vmatrix} = (-1)\,[(6) - (6)] = 0 \\
c_{22} &= (-1)^{2+2} \begin{vmatrix} 1 & -1 \\ 4 & -3 \end{vmatrix} = (1)\,[(-3) - (-4)] = 1 \\
c_{23} &= (-1)^{2+3} \begin{vmatrix} 1 & -2 \\ 4 & -6 \end{vmatrix} = (-1)\,[(-6) - (-8)] = -2 \\
c_{31} &= (-1)^{3+1} \begin{vmatrix} -2 & -1 \\ 7 & 3 \end{vmatrix} = (1)\,[(-6) - (-7)] = 1 \\
c_{32} &= (-1)^{3+2} \begin{vmatrix} 1 & -1 \\ -3 & 3 \end{vmatrix} = (-1)\,[(3) - (-3)] = 0 \\
c_{33} &= (-1)^{3+3} \begin{vmatrix} 1 & -2 \\ -3 & 7 \end{vmatrix} = (1)\,[(7) - (6)] = 1
\end{aligned}$$

The cofactor matrix is given by

$$A_{ij} = \begin{bmatrix} -3 & 3 & -10 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$$

The adjoint matrix is given by

$$adj(A) = A^{+} = A'_{ij} = \begin{bmatrix} -3 & 0 & 1 \\ 3 & 1 & 0 \\ -10 & 2 & 1 \end{bmatrix}$$

The inverse is given by

$$\frac{adj(A)}{|A|} = \frac{adj(A)}{1} = \frac{\begin{bmatrix} -3 & 0 & 1 \\ 3 & 1 & 0 \\ -10 & 2 & 1 \end{bmatrix}}{1} = \begin{bmatrix} -3 & 0 & 1 \\ 3 & 1 & 0 \\ -10 & 2 & 1 \end{bmatrix}$$

d. Solve the equation Ax=b using the inverse you found in part 2c

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -3 & 7 & 3 \\ 4 & -6 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ -8 \\ 7 \end{bmatrix}$$

$$\begin{aligned} x =& A^{-1}b \\ =& \begin{bmatrix} -3 & 0 & 1 \\ 3 & 1 & 0 \\ -10 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -8 \\ 7 \end{bmatrix} = \begin{bmatrix} -6+7 \\ 6-8 \\ -20+16+7 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \end{aligned}$$

e. Solve the equation Ax=b using Cramer's rule.

Hint: $\begin{vmatrix} 2 & -2 & -1 \\ -8 & 7 & 3 \\ 7 & -6 & -3 \end{vmatrix} = 1$

So $x_1 = \frac{1}{1} = 1$.

$$\begin{aligned} x_2 =& \frac{\begin{vmatrix} 1 & 2 & -1 \\ -3 & -8 & 3 \\ 4 & 7 & -3 \end{vmatrix}}{1} \\ =& \frac{((1 \times -8 \times -3) + (2 \times 3 \times 4) + (-1 \times -3 \times 7)) - ((-1 \times -8 \times 4) + (2 \times -3 \times -3) + (1 \times 3 \times 7))}{1} \\ =& (24 + 24 + 21) - (32 + 18 + 21) \\ =& 69 - 71 = -2 \end{aligned}$$

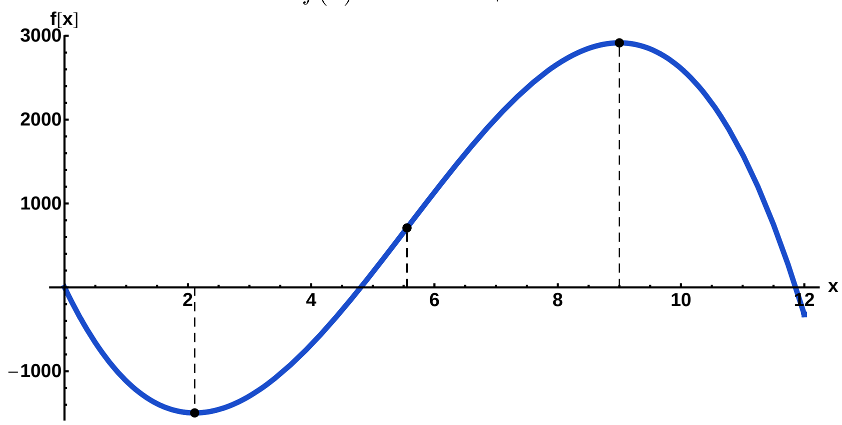
$$\begin{aligned} x_3 =& \frac{\begin{vmatrix} 1 & -2 & 2 \\ -3 & 7 & -8 \\ 4 & -6 & 7 \end{vmatrix}}{1} \\ =& \frac{((1 \times 7 \times 7) + (-2 \times -8 \times 4) + (2 \times -3 \times -6)) - ((2 \times 7 \times 4) + (-2 \times -3 \times 7) + (1 \times -8 \times -6))}{1} \\ =& (49 + 64 + 36) - (56 + 42 + 48) \\ =& 149 - 146 = 3 \end{aligned}$$

Problem 3. [15 Points]

For the following problem, find the critical points. For each critical point state whether the function is at a relative maximum, relative minimum, or otherwise. Check to see if there are potential points of inflection **at points other than** critical points.

$$f(x) = -27x^3 + 450x^2 - 1539x$$

FIGURE 1. $f(x) = -27x^3 + 450x^2 - 1539x$



Find the first and second derivatives of f.

$$\begin{aligned} f(x) &= -27x^3 + 450x^2 - 1539x \\ \frac{df(x)}{dx} &= -81x^2 + 900x - 1539 \\ \frac{d^2f(x)}{dx^2} &= -162x + 900 \end{aligned} \tag{3.1}$$

Set the first derivative of f equal to zero and solve for the critical values of x.

$$\begin{aligned} \frac{df(x)}{dx} &= -81x^2 + 900x - 1539 = 0 \\ &\Rightarrow -9(9x^2 - 100x + 171) = 0 \\ &\Rightarrow (9x - 19)(x - 9) = 0 \\ \Rightarrow (9x - 19) &= 0 \text{ or } (x - 9) = 0 \\ \Rightarrow x &= \frac{19}{9} \text{ or } x = 9 \end{aligned} \tag{3.2}$$

Now evaluate the second derivative of f(x) at each critical value of x. First for $x = \frac{19}{9}$

$$\begin{aligned}\frac{d^2 f(x)}{dx^2} &= -162x + 900 \\ \frac{d^2 f(x)}{dx^2}\left(\frac{19}{9}\right) &= -162\left(\frac{19}{9}\right) + 900 \\ &= -342 + 900 = 558 \\ &\Rightarrow x = \frac{19}{9} \text{ is a minimum}\end{aligned} \tag{3.3}$$

Now for $x = 9$

$$\begin{aligned}\frac{d^2 f(x)}{dx^2} &= -162x + 900 \\ \frac{d^2 f(x)}{dx^2}(9) &= -162(9) + 900 \\ &= -1458 + 900 = -558 \\ &\Rightarrow x = 9 \text{ is a maximum}\end{aligned} \tag{3.4}$$

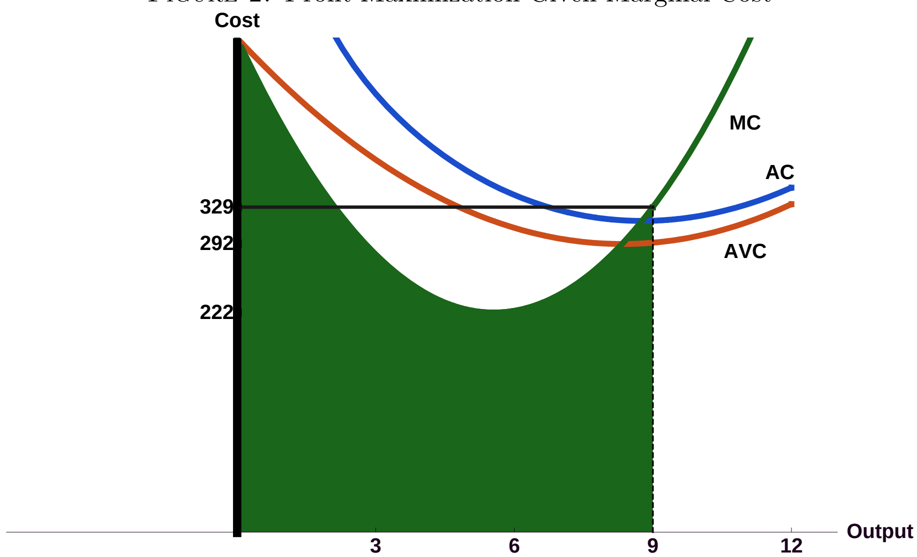
We obtain the inflection point by setting the second derivative equal to zero.

$$\begin{aligned}\frac{d^2 f(x)}{dx^2} &= -162x + 900 = 0 \\ &\Rightarrow -162x = -900 \\ &\Rightarrow x = \frac{900}{162} = \frac{100}{18} = \frac{50}{9}\end{aligned} \tag{3.5}$$

Problem 4. [20 Points]

The cost function for a firm is a rule or mapping that tells the minimum total cost of production of any output level produced by the firm for a fixed level of input prices. If the variable y represents the output of the firm, then the cost function is given by c(y,w) or c(y). Marginal cost represents the change in the cost of production for the firm as output changes and is given by the derivative of the cost function with respect to output, i.e., Marginal Cost(MC) = $\frac{dc(y)}{dy}$. A graph of price and marginal cost is given in figure 2.

FIGURE 2. Profit Maximization Given Marginal Cost



a. Consider a competitive firm with the following technology (as represented by its cost function) and output price.

$$price \ = \ p \ = \$329$$

$$cost \ = \ c(y) \ = 200 \ + \ 500y \ - \ 50y^2 \ + \ 3y^3$$

Without writing down an equation for profit, find the levels of output which potentially maximize profit using what you have learned in general about profit maximization for a competitive firm.

We set marginal cost equal price to and solve for y.

$$500 - 100y + 9y^2 = 329$$

$$\Rightarrow 9y^2 - 100y + 171 = 0$$

$$\Rightarrow (9y - 19)(y - 9) = 0$$

$$\Rightarrow y \ = \ \frac{19}{9} \text{ or } y = 9$$

b. Given that you have no second order conditions from profit maximization per se, make a coherent argument about which of the two potential output levels you found in part 4a maximizes profit.

At the point $\frac{19}{9}$ p is equal to marginal cost and the firm makes no profit on this particular unit. If the firm reduces output from the point $y = \frac{19}{9}$ one can see from figure 2 that marginal cost would be higher than price and this would be a bad idea. If the firm were to increase output from $y = \frac{19}{9}$ one can see from figure 2 that price is higher than marginal cost and so the firm should increase output and obtain some profit on at least the first unit beyond $\frac{19}{9}$. So $y = \frac{19}{9}$ is not a maximum profit point.

Now consider the other point where price = MC, i.e., where $y = 9$. If the firm were to increase output beyond $y = 9$, can see from figure 2 that marginal cost would be higher than price and this would be a bad idea. If the firm decreases output from $y = 9$, one can see that price is higher than marginal cost and the firm should increase output back up to $y = 9$.

c. Explain in words why setting price equal to marginal cost and solving for the optimal output y gives the same answers as taking the derivative of profit with respect to y, setting the result equal to zero and solving for the optimal y. Remember that

$$Profit = py - c(y)$$

$$Profit = 343y - \left[200 + 500y - 50\, y^2 + 3y^3\right]$$

Maximizing profit we obtain

$$\frac{d\,Profit}{dy} = p - \frac{d\,c}{dy} = 0 \tag{4.1a}$$

$$\Rightarrow p = \frac{d\,c}{dy} = MC(y) \tag{4.1b}$$

Thus the price = MC condition comes from maximizing the function $py - c(y)$.

Note: Here is the interpretation of the p = MC condition which you were not asked to provide.

The right hand side of equation 4.1b is the cost of producing one more unit of output or the marginal cost of output, MC(y). The left hand side of equation 4.1b is the marginal revenue from selling one more unit of that output. For a competitive firm the marginal revenue from selling one more unit of output is just the price of that output. If the revenue from selling one more unit of an output is greater than its cost of production, i.e. p $\geq$ MC(y), we should continue producing more units of the output up until the point where p = MC(y)

d. Show that variable cost for this firm when it maximizes profit with a price of \$329 is \$2637.

$$\begin{aligned} VC(y) &= 200 \ + \ 500y \ - \ 50y^2 \ + \ 3y^3 \\ &=(500)(9) - (50)(81) + (3)(729) \\ &=4500 - 4050 + 2187 = 2637 \end{aligned}$$

e. Given that revenue for this profit maximizing firm is \$2961, what is producer surplus?
PS = 2961 - 2637 = \$324.

f. Cross-hatch this level of producer surplus in Figure 2.