Some Hints:

(1) Allocate your time efficiently. Spend time on problems in proportion to points possible.
(2) \(2^6 = 4^3 = 64 \Rightarrow 64^{1/3} = 4\)
(3) \(2^{12} = 2^6 \times 2^6 = 64^2 = 4096\)
(4) \(2^5 \times 2^6 = 2^{11} = 2048\)
(5) \(2^{11} \times 3 = 6144\)
(6) \(16^{9/4} = 2^9\)

For each of the first three problems, write an equation that represents profit as a function of the two inputs \(x_1\) and \(x_2\). Find levels of \(x_1\) and \(x_2\) that potentially maximize profit. Show that the levels of \(x_1\) and \(x_2\) you find actually maximize profit by checking second order conditions.
Problem 1 (25 Points).

a. What is the profit equation for the following firm

\[ f(x_1, x_2) = 10x_1 + 30x_2 - x_1^2 + x_1x_2 - 2x_2^2 \]

\[ p = 3 \]

\[ w_1 = 9, \quad w_2 = 6 \]

\[ \pi = 3(10x_1 + 30x_2 - x_1^2 + x_1x_2 - 2x_2^2) - 9x_1 - 6x_2 \]

\[ = 30x_1 + 90x_2 - 3x_1^2 + 3x_1x_2 - 6x_2^2 - 9x_1 - 6x_2 \]

\[ = 21x_1 + 84x_2 - 3x_1^2 + 3x_1x_2 - 6x_2^2 \]

b. Fill in the following table of derivatives of the profit equation.

<table>
<thead>
<tr>
<th>[ \frac{\partial \pi}{\partial x_1} ]</th>
<th>[ \frac{\partial \pi}{\partial x_2} ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ 21 - 6x_1 + 3x_2 ]</td>
<td>[ 84 + 3x_1 - 12x_2 ]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>[ \frac{\partial^2 \pi}{\partial x_1 \partial x_1} ]</th>
<th>[ \frac{\partial^2 \pi}{\partial x_1 \partial x_2} ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ -6 ]</td>
<td>[ 3 ]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>[ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} ]</th>
<th>[ \frac{\partial^2 \pi}{\partial x_2 \partial x_2} ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ 3 ]</td>
<td>[ -12 ]</td>
</tr>
</tbody>
</table>
c. Find potential profit maximizing levels of \( x_1 \) and \( x_2 \).

To do so, write the two first order conditions as a system of equations after moving the constant to the right hand side.

\[
-6x_1 + 3x_2 = -21 \tag{1.1a}
\]
\[
3x_1 - 12x_2 = -84 \tag{1.1b}
\]

Multiply equation 1.1a by 4 and then add it to the second equation. First multiply

\[
4 \times (-6x_1 + 3x_2 = -21) \Rightarrow -24x_1 + 12x_2 = -84 \tag{1.2}
\]

Then add this new equation to equation 1.1b as follows.

\[
-24x_1 + 12x_2 = -84 \tag{1.3}
\]

Now substitute \( x_1 = 8 \) into equation 1.1a and solve for \( x_2 \) as follows.

\[
-6 \times 8 + 3x_2 = -21 \Rightarrow -48 + 3x_2 = -21 \Rightarrow 3x_2 = 27 \Rightarrow x_2 = \frac{27}{3} = 9 \tag{1.4}
\]
d. By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of $x_1$ and $x_2$.

The Hessian is given by

$$
\begin{bmatrix}
\frac{\partial^2 \pi}{\partial x_1 \partial x_1} &= -6 \\
\frac{\partial^2 \pi}{\partial x_1 \partial x_2} &= 3 \\
\frac{\partial^2 \pi}{\partial x_2 \partial x_1} &= 3 \\
\frac{\partial^2 \pi}{\partial x_2 \partial x_2} &= -12
\end{bmatrix}
$$

The determinant of the Hessian is given by

$$
\text{det}[H] = ((-6)(-12)) - ((3)(3)) = 72 - 9 = 63
$$

Both diagonal elements of the Hessian are negative and the determinant is positive so the know that the $x_1 = 8$ and $x_2 = 9$ is a point of maximal profit.
Problem 2 (25 Points).

a. What is the profit equation for the following firm

\[ f(x_1, x_2) = 100x_1 + 50x_2 - 2x_1^2 + x_1x_2 - 2x_2^2 \]

\[ p = 2 \]

\[ w_1 = 90, \quad w_2 = 60 \]

\[ x_1 = 16 \]

\[ \pi = 2(100x_1 + 50x_2 - 2x_1^2 + x_1x_2 - 2x_2^2) - 90x_1 - 60x_2 \]

\[ = 200x_1 + 100x_2 - 4x_1^2 + 2x_1x_2 - 4x_2^2 - 90x_1 - 60x_2 \]

\[ = 110x_1 + 40x_2 - 4x_1^2 + 2x_1x_2 - 4x_2^2 \]

b. Fill in the following table of derivatives of the profit equation.

<table>
<thead>
<tr>
<th>( \frac{\partial \pi}{\partial x_1} = 110 - 8x_1 + 2x_2 )</th>
<th>( \frac{\partial \pi}{\partial x_2} = 40 + 2x_1 - 8x_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -8 )</td>
<td>( \frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 2 )</td>
</tr>
<tr>
<td>( \frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 2 )</td>
<td>( \frac{\partial^2 \pi}{\partial x_2 \partial x_2} = 8 )</td>
</tr>
</tbody>
</table>
c. Find potential profit maximizing levels of \( x_1 \) and \( x_2 \).

To do so, write the two first order conditions as a system of equations.

\[
\begin{align*}
-8x_1 + 2x_2 &= -110 \\
2x_1 - 8x_2 &= -40
\end{align*}
\]

Multiply equation 2.1b by 4 and then add it equation 2.1a. First multiply

\[
4 \times (2x_1 - 8x_2 = -40) \Rightarrow 8x_1 - 32x_2 = -160
\]

Then add this new equation to equation 2.1a as follows.

\[
\begin{align*}
8x_1 - 32x_2 &= -160 \\
-8x_1 + 2x_2 &= -110 \\
\hline
x_1 - 30x_2 &= -270 \\
\Rightarrow \quad x_2 &= \frac{-270}{-30} = 9
\end{align*}
\]

Now substitute \( x_2 = 9 \) into equation 2.1a and solve for \( x_1 \) as follows.

\[
\begin{align*}
-8x_1 + 2x_2 &= -110 \\
\Rightarrow -8x_1 + 2(9) &= -110 \\
\Rightarrow -8x_1 + 18 &= -110 \\
\Rightarrow 8x_1 &= 128 \\
\Rightarrow x_1 &= \frac{128}{8} = 16
\end{align*}
\]
d. By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of \( x_1 \) and \( x_2 \).

The Hessian is given by

\[
H = \begin{pmatrix}
\frac{\partial^2 \pi}{\partial x_1 \partial x_1} &=& -8 \\
\frac{\partial^2 \pi}{\partial x_1 \partial x_2} &=& 2 \\
\frac{\partial^2 \pi}{\partial x_2 \partial x_1} &=& 2 \\
\frac{\partial^2 \pi}{\partial x_2 \partial x_2} &=& 8
\end{pmatrix}
\]

The determinant of the Hessian is given by

\[
\det[H] = ((-8)(-8)) - ((2)(2)) = 64 - 4 = 60
\]

Both diagonal elements of the Hessian are negative and the determinant is positive so the know that the \( x_1 = 16 \) and \( x_2 = 9 \) is a point of maximal profit.
Problem 3 (35 Points).

a. What is the profit equation for the following firm

\[ f(x_1, x_2) = x_1^{1/4} x_2^{1/3} \]
\[ p = 24 \]
\[ w_1 = 3, \quad w_2 = 1 \]
\[ x_1 = 16 \]
\[ \pi = 24 \left( x_1^{1/4} x_2^{1/3} \right) - 3x_1 - x_2 \]
\[ = 24x_1^{1/4} x_2^{1/3} - 3x_1 - x_2 \]

b. Fill in the following table of derivatives of the profit equation.

<table>
<thead>
<tr>
<th>( \frac{\partial \pi}{\partial x_1} )</th>
<th>( \frac{\partial \pi}{\partial x_2} )</th>
<th>( \frac{\partial^2 \pi}{\partial x_1 \partial x_1} )</th>
<th>( \frac{\partial^2 \pi}{\partial x_1 \partial x_2} )</th>
<th>( \frac{\partial^2 \pi}{\partial x_2 \partial x_1} )</th>
<th>( \frac{\partial^2 \pi}{\partial x_2 \partial x_2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 6x_1^{-3/4} x_2^{-1/3} - 3 )</td>
<td>( 8x_1^{1/4} x_2^{-2/3} - 1 )</td>
<td>( 9x_1^{-5/4} x_2^{-1/3} )</td>
<td>( 2x_1^{-3/4} x_2^{-2/3} )</td>
<td>( 2x_1^{-3/4} x_2^{-2/3} )</td>
<td>( -16x_1^{-1/4} x_2^{-5/3} )</td>
</tr>
</tbody>
</table>
c. Find potential profit maximizing levels of $x_1$ and $x_2$.

To do this, solve the following systems of equations from the first order conditions for $x_1$ and $x_2$.

\[
\begin{align*}
6x_1^{-3/4}x_2^{1/3} - 3 &= 0 \quad (3.1) \\
8x_1^{1/4}x_2^{-2/3} - 1 &= 0 \quad (3.2)
\end{align*}
\]

Rearrange the first equation in 3.1 to obtain

\[
x_1^{-3/4}x_2^{1/3} = \frac{3}{6} = \frac{1}{2}
\]

\[
\Rightarrow x_1^{3/4}x_2^{-1/3} = \frac{1}{2}x_2^{3/4}
\]

\[
\Rightarrow x_2 = \left(\frac{1}{2}\right)^{3/4} x_1^{3/4}
\]

Rearrange the second equation 3.2 slightly to obtain

\[
x_1^{1/4}x_2^{-2/3} = \frac{1}{8} \quad (3.2')
\]

Now substitute $x_2$ from equation 3.1.a into equation 3.2' to obtain
\[ x_{1}^{1/4} \left( \frac{1}{2} \right)^{3} x_{1}^{9/4} \left( x_{1}^{1/4} \right)^{-2/3} = \frac{1}{8} \]

\[ \Rightarrow x_{1}^{1/4} \left( \frac{1}{2} \right)^{-2} x_{1}^{-3/2} = \frac{1}{8} \]

\[ \Rightarrow x_{1}^{-5/4} \left( \frac{1}{2} \right)^{-2} = \frac{1}{8} \]

\[ \Rightarrow x_{1}^{-5/4} = \frac{1}{8} \left( \frac{1}{2} \right)^{2} \]

\[ \Rightarrow x_{1} = \left( \frac{1}{8} \left( \frac{1}{2} \right)^{2} \right)^{-4/5} \]

\[ = \left( \frac{1}{8} \right)^{-4/5} \left( \frac{1}{2} \right)^{-8/5} \]

\[ = \left( 2^{-3} \right)^{-4/5} 2^{8/5} \]

\[ = 2^{12/5} 2^{8/5} \]

\[ = 2^{20/5} = 2^{4} = 16 \]

Now substitute \( x_{1} \) from equation 3.2.a into equation 3.1.a to obtain

\[ x_{2} = \left( \frac{1}{2} \right)^{3} x_{1}^{9/4} \]

\[ = \left( \frac{1}{2} \right)^{3} (16)^{9/4} \]

\[ = 2^{-3} 2^{9} \]

\[ = 2^{6} = 64 \]
d. In this table fill in values of $x_1$ and $x_2$ given to obtain numerical answers for the Hessian matrix.

<table>
<thead>
<tr>
<th></th>
<th>$\frac{\partial^2 \pi}{\partial x_1 \partial x_1}$</th>
<th>$\frac{\partial^2 \pi}{\partial x_1 \partial x_2}$</th>
<th>$\frac{\partial^2 \pi}{\partial x_2 \partial x_1}$</th>
<th>$\frac{\partial^2 \pi}{\partial x_2 \partial x_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculated</td>
<td>$-\frac{9}{2} \frac{1}{x_1} - \frac{7}{4} \frac{1}{x_1^2}$</td>
<td>$2x_1^{3/4} x_2^{-2/3}$</td>
<td>$2x_1^{-5/4} x_2^{1/3}$</td>
<td>$- \frac{16}{3} x_1^{1/4} x_2^{-5/4}$</td>
</tr>
</tbody>
</table>

\[
\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -\frac{9}{2} (16)^{-7/4}(64)^{1/3} = (-1)(9)2^{-1}(2^4)^{-7/4}(2^6)^{1/3} = (-1)(9)2^{-12}7(2^2) = (-9)2^{-6} = - \frac{9}{64}
\]

\[
\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = \frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 2(16)^{-3/4}(64)^{-2/3} = 2(2^4)^{-3/4}(2^6)^{-2/3} = (2)2^{-3}2^{-4} = 2^{-6} = \frac{1}{64}
\]

\[
\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = \frac{16}{3} (16)^{1/4}(64)^{-5/3} = \frac{24}{3} (2^4)^{1/4}(2^6)^{-5/3} = (-1)(2^4)3^{-1}(2)2^{-30/3} = (-1)3^{-1}2^{5}2^{-10} = \frac{-1}{3 \times 2^5} = - \frac{1}{3 \times 32} = - \frac{1}{96}
\]

This then gives for the Hessian

H =

<table>
<thead>
<tr>
<th></th>
<th>$\frac{\partial^2 \pi}{\partial x_1 \partial x_1}$</th>
<th>$\frac{\partial^2 \pi}{\partial x_1 \partial x_2}$</th>
<th>$\frac{\partial^2 \pi}{\partial x_2 \partial x_1}$</th>
<th>$\frac{\partial^2 \pi}{\partial x_2 \partial x_2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculated</td>
<td>$\frac{-9}{64}$</td>
<td>$\frac{1}{64}$</td>
<td>$\frac{-1}{96}$</td>
<td>$\frac{-1}{96}$</td>
</tr>
</tbody>
</table>
e. By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of $x_1$ and $x_2$.

\[
H = \begin{bmatrix}
\frac{\partial^2 \pi}{\partial x_1 \partial x_1} &= -\frac{9}{64} \\
\frac{\partial^2 \pi}{\partial x_1 \partial x_2} &= \frac{1}{64} \\
\frac{\partial^2 \pi}{\partial x_2 \partial x_1} &= \frac{1}{64} \\
\frac{\partial^2 \pi}{\partial x_2 \partial x_2} &= -\frac{1}{96}
\end{bmatrix}
\]

The determinant of the Hessian is given by

\[
\text{det}[H] = \left( -\frac{9}{64} \right) \left( -\frac{1}{96} \right) - \left( \frac{1}{64} \right) \left( \frac{1}{64} \right)
\]

\[
= \left( -\frac{9}{2^6} \right) \left( -\frac{1}{2^5 \times 3} \right) - \left( \frac{1}{2^6} \right) \left( \frac{1}{2^6} \right)
\]

\[
= \left( \frac{9 \times 2}{2^6 \times 2^5 \times 3} \right) - \left( \frac{1}{2^{12}} \right)
\]

\[
= \left( \frac{18}{2^{11} \times 3} \right) - \left( \frac{3}{2^{12} \times 3} \right)
\]

\[
= \frac{15}{2^{12} \times 3}
\]

\[
= \frac{5}{2^{12}} = \frac{5}{4096}
\]

Given that the diagonals of the Hessian are negative and the determinant is positive, we know that the values of $x$ we found represent a maximum.
### Problem 4 (15 Points)
Find the listed partial derivatives of each of the following function.

\[ \mathcal{L}(x_1, x_2, \lambda) = (10x_1 + 30x_2 - x_1^2 + x_1x_2 - 2x_2^2) - \lambda(9x_1 + 6x_2 - 126) \]

<table>
<thead>
<tr>
<th>Partial Derivative</th>
<th>Calculation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} )</td>
<td>10 - 2x_1 + x_2 - 9\lambda</td>
</tr>
<tr>
<td>( \frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} )</td>
<td>30 + x_1 - 4x_2 - 6\lambda</td>
</tr>
<tr>
<td>( \frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} )</td>
<td>-(9x_1 + 6x_2 - 126)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Second Partial Derivative</th>
<th>Calculation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} )</td>
<td>-2</td>
</tr>
<tr>
<td>( \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} )</td>
<td>1</td>
</tr>
<tr>
<td>( \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} )</td>
<td>9</td>
</tr>
<tr>
<td>( \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} )</td>
<td>1</td>
</tr>
<tr>
<td>( \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} )</td>
<td>-4</td>
</tr>
<tr>
<td>( \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} )</td>
<td>6</td>
</tr>
<tr>
<td>( \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} )</td>
<td>0</td>
</tr>
</tbody>
</table>