ECONOMICS 207
SPRING 2008
EXAM 5
KEY

Some Hints:

(1) Allocate your time efficiently. Spend time on problems in proportion to points possible.
(2) $2^6 = 4^3 = 64 \Rightarrow 64^{1/3} = 4$
(3) $2^{12} = 2^6 \times 2^6 = 64^2 = 4096$
(4) $2^5 \times 2^6 = 2^{11} = 2048$
(5) $2^{11} \times 3 = 6144$
(6) $16^{9/4} = 2^9$

For each of the first three problems, write an equation that represents profit as a function of the two inputs x_1 and x_2. Find levels of of x_1 and x_2 that potentially maximize profit. Show that the levels of x_1 and x_2 you find actually maximize profit by checking second order conditions.

Date: April 23, 2008.

Problem 1 (25 Points)**.**

a. What is the profit equation for the following firm

$$\begin{aligned} f(x_1, x_2) =& 10x_1 + 30x_2 - x_1^2 + x_1x_2 - 2x_2^2 \\ p =& \ 3 \\ w_1 =& \ 9, \qquad w_2 = 6 \end{aligned}$$

$$\begin{aligned} \pi =& 3(10x_1 + 30x_2 - x_1^2 + x_1x_2 - 2x_2^2) - 9x_1 - 6x_2 \\ =& 30x_1 + 90x_2 - 3x_1^2 + 3x_1x_2 - 6x_2^2 - 9x_1 - 6x_2 \\ =& 21x_1 + 84x_2 - 3x_1^2 + 3x_1x_2 - 6x_2^2 \end{aligned}$$

b. Fill in the following table of derivatives of the profit equation.

$\frac{\partial \pi}{\partial x_1} = 21 - 6x_1 + 3x_2$	$\frac{\partial \pi}{\partial x_2} = 84 + 3x_1 - 12x_2$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -6$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 3$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 3$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} =$ -12

c. Find potential profit maximizing levels of x_1 and x_2.

To do so, write the two first order conditions as a system of equations after moving the constant to the right hand side.

$$-6x_1 + 3x_2 = -21 \tag{1.1a}$$

$$3x_1 - 12x_2 = -84 \tag{1.1b}$$

Multiply equation 1.1a by 4 and then add it to the second equation. First multiply

$$\begin{aligned} &4\times(-6x_1 + 3x_2 = -21) \\ &\Rightarrow -24x_1 + 12x_2 = -84 \end{aligned} \tag{1.2}$$

Then add this new equation to equation 1.1b as follows.

$$\begin{aligned} -24x_1 + 12x_2 &= -84 \\ 3x_1 - 12x_2 &= -84 \\ \hline -21x_1 \qquad &= -168 \\ \Rightarrow \quad x_1 &= \frac{-168}{-21} = 8 \end{aligned} \tag{1.3}$$

Now substitute $x_1 = 8$ into equation 1.1a and solve for x_2 as follows.

$$\begin{aligned} -6x_1 + 3x_2 &= -21 \\ \Rightarrow -6(8) + 3x_2 &= -21 \\ \Rightarrow -48 + 3x_2 &= -21 \\ \Rightarrow 3x_2 &= 27 \\ \Rightarrow x_2 &= \frac{27}{3} = 9 \end{aligned} \tag{1.4}$$

d. By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

The Hessian is given by

H =

$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -6$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 3$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 3$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -12$

The determinant of the Hessian is given by

$$\det[H] = ((-6)(-12)) - ((3)(3))$$
$$= 72 - 9 = 63$$

Both diagonal elements of the Hessian are negative and the determinant is positive so the know that the $x_1 = 8$ and $x_2 = 9$ is a point of maximal profit.

Problem 2 (25 Points)**.**

a. What is the profit equation for the following firm

$$\begin{aligned} f(x_1, x_2) =& 100x_1 + 50x_2 - 2x_1^2 + x_1x_2 - 2x_2^2 \\ p =& 2 \\ w_1 =& 90, \qquad w_2 = 60 \\ x_1 =& 16 \end{aligned}$$

$$\begin{aligned} \pi =& 2(100x_1 + 50x_2 - 2x_1^2 + x_1x_2 - 2x_2^2) - 90x_1 - 60x_2 \\ =& 200x_1 + 100x_2 - 4x_1^2 + 2x_1x_2 - 4x_2^2 - 90x_1 - 60x_2 \\ =& 110x_1 + 40x_2 - 4x_1^2 + 2x_1x_2 - 4x_2^2 \end{aligned}$$

b. Fill in the following table of derivatives of the profit equation.

$\frac{\partial\pi}{\partial x_1} = 110 - 8x_1 + 2x_2$	$\frac{\partial\pi}{\partial x_2} = 40 + 2x_1 - 8x_2$
$\frac{\partial^2\pi}{\partial x_1\partial x_1} = -8$	$\frac{\partial^2\pi}{\partial x_1\partial x_2} = 2$
$\frac{\partial^2\pi}{\partial x_2\partial x_1} = 2$	$\frac{\partial^2\pi}{\partial x_2\partial x_2} - 8$

c. Find potential profit maximizing levels of x_1 and x_2.

To do so, write the two first order conditions as a system of equations.

$$-8x_1 + 2x_2 = -110 \tag{2.1a}$$

$$2x_1 - 8x_2 = -40 \tag{2.1b}$$

Multiply equation 2.1b by 4 and then add it equation 2.1a. First multiply

$$\begin{aligned} 4 \times (2x_1 \; - 8x_2 &= -40) \\ \Rightarrow \; 8x_1 - 32x_2 &= -160 \end{aligned} \tag{2.2}$$

Then add this new equation to equation 2.1a as follows.

$$\begin{aligned} 8x_1 \; - 32x_2 &= -160 \\ -8x_1 \; + 2x_2 &= -110 \\ \hline x_1 \; - 30x_2 &= -270 \\ \Rightarrow \qquad x_2 &= \frac{-270}{-30} = 9 \end{aligned} \tag{2.3}$$

Now substitute $x_2 = 9$ into equation 2.1a and solve for x_1 as follows.

$$\begin{aligned} -8x_1 + 2x_2 &= -110 \\ \Rightarrow -8x_1 + 2(9) &= -110 \\ \Rightarrow -8x_1 + 18 &= -110 \\ \Rightarrow 8x_1 &= 128 \\ \Rightarrow x_1 &= \frac{128}{8} = 16 \end{aligned} \tag{2.4}$$

d. By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

The Hessian is given by

H =

$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -8$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 2$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 2$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} - 8$

The determinant of the Hessian is given by

$$\det[H] = ((-8)(-8)) - ((2)(2))$$
$$= 64 - 4 = 60$$

Both diagonal elements of the Hessian are negative and the determinant is positive so the know that the $x_1 = 16$ and $x_2 = 9$ is a point of maximal profit.

Problem 3 (35 Points).

a. What is the profit equation for the following firm

$$
\begin{aligned}
f(x_1, x_2) &= x_1^{1/4} x_2^{1/3} \\
p &= 24 \\
w_1 &= 3, \qquad w_2 = 1 \\
x_1 &= 16
\end{aligned}
$$

$$
\begin{aligned}
\pi =& 24\left(x_1^{1/4} x_2^{1/3}\right) - 3x_1 - x_2 \\
=& 24 x_1^{1/4} x_2^{1/3} - 3x_1 - x_2
\end{aligned}
$$

b. Fill in the following table of derivatives of the profit equation.

$\frac{\partial \pi}{\partial x_1} = 6x_1^{-3/4} x_2^{1/3} - 3$	$\frac{\partial \pi}{\partial x_2} = 8x_1^{1/4} x_2^{-2/3} - 1$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -\frac{9}{2} x_1^{-7/4} x_2^{1/3}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 2x_1^{-3/4} x_2^{-2/3}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 2x_1^{-3/4} x_2^{-2/3}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -\frac{16}{3} x_1^{1/4} x_2^{-5/3}$

c. Find potential profit maximizing levels of x_1 and x_2.

To do this, solve the following systems of equations from the first order conditions for x_1 and x_2.

$$6x_1^{-3/4}x_2^{1/3} - 3 = 0 \tag{3.1}$$

$$8x_1^{1/4}x_2^{-2/3} - 1 = 0 \tag{3.2}$$

Rearrange the first equation in 3.1 to obtain

$$\begin{aligned} x_1^{-3/4}x_2^{1/3} &= \frac{3}{6} = \frac{1}{2} \\ \Rightarrow x_1^{3/4}x_1^{-3/4}x_2^{1/3} &= \frac{1}{2}x_1^{3/4} \\ \Rightarrow x_2^{1/3} &= \frac{1}{2}x_1^{3/4} \qquad (3.1.a) \\ \Rightarrow x_2 &= \left(\frac{1}{2}\right)^3 \left(x_1^{3/4}\right)^3 \\ &= \left(\frac{1}{2}\right)^3 x_1^{9/4} \end{aligned}$$

Rearrange the second equation 3.2 slightly to obtain

$$x_1^{1/4}x_2^{-2/3} = \frac{1}{8} \tag{3.2'}$$

Now substitute x_2 from equation 3.1.a into equation 3.2′ to obtain

$$
\begin{aligned}
x_1^{1/4}\left(\left(\frac{1}{2}\right)^3 x_1^{9/4}\right)^{-2/3} &= \frac{1}{8} \\
\Rightarrow x_1^{1/4}\left(\frac{1}{2}\right)^{-2} x_1^{-3/2} &= \frac{1}{8} \\
\Rightarrow x_1^{-5/4}\left(\frac{1}{2}\right)^{-2} &= \frac{1}{8} \\
\Rightarrow x_1^{-5/4} &= \frac{1}{8}\left(\frac{1}{2}\right)^{2} \\
\Rightarrow x_1 &= \left(\frac{1}{8}\left(\frac{1}{2}\right)^{2}\right)^{-4/5} \\
&= \left(\frac{1}{8}\right)^{-4/5}\left(\frac{1}{2}\right)^{-8/5} \\
&= (2^{-3})^{-4/5} 2^{8/5} \\
&= 2^{12/5} 2^{8/5} \\
&= 2^{20/5} = 2^4 = 16
\end{aligned} \tag{3.2.a}
$$

Now substitute x_1 from equation 3.2.a into equation 3.1.a to obtain

$$
\begin{aligned}
x_2 &= \left(\frac{1}{2}\right)^3 x_1^{9/4} \\
&= \left(\frac{1}{2}\right)^3 (16)^{9/4} \\
&= 2^{-3} 2^{9} \\
&= 2^6 = 64
\end{aligned}
$$

d. In this table fill in values of x_1 and x_2 given to obtain numerical answers for the Hessian matrix.

H =

$\frac{\partial^2\pi}{\partial x_1 \partial x_1} = -\frac{9}{2} x_1^{-7/4} x_2^{1/3}$	$\frac{\partial^2\pi}{\partial x_1 \partial x_2} = 2 x_1^{-3/4} x_2^{-2/3}$
$\frac{\partial^2\pi}{\partial x_2 \partial x_1} = 2 x_1^{-3/4} x_2^{-2/3}$	$\frac{\partial^2\pi}{\partial x_2 \partial x_2} = -\frac{16}{3} x_1^{1/4} x_2^{-5/3}$

$$\frac{\partial^2\pi}{\partial x_1 \partial x_1} = -\frac{9}{2}(16)^{-7/4}(64)^{1/3} = (-1)(9)2^{-1}(2^4)^{-7/4}(2^6)^{1/3}$$

$$= (-1)(9)2^{-1}2^{-7}(2^2) = (-9)2^{-6} = -\frac{9}{64}$$

$$\frac{\partial^2\pi}{\partial x_2 \partial x_1} = \frac{\partial^2\pi}{\partial x_1 \partial x_2} = 2(16)^{-3/4}(64)^{-2/3} = 2(2^4)^{-3/4}(2^6)^{-2/3}$$

$$= (2)2^{-3}2^{-4} = 2^{-6} = \frac{1}{64}$$

$$\frac{\partial^2\pi}{\partial x_2 \partial x_2} = -\frac{16}{3}(16)^{1/4}(64)^{-5/3} = -\frac{2^4}{3}\left(2^4\right)^{1/4}\left(2^6\right)^{-5/3}$$

$$= (-1)(2^4)3^{-1}(2)2^{-30/3}$$

$$= (-1)3^{-1}2^5 2^{-10} = \frac{-1}{3 \times 2^5} = -\frac{1}{3 \times 32} = -\frac{1}{96}$$

This then gives for the Hessian

H =

$\frac{\partial^2\pi}{\partial x_1 \partial x_1} = \frac{-9}{64}$	$\frac{\partial^2\pi}{\partial x_1 \partial x_2} = \frac{1}{64}$
$\frac{\partial^2\pi}{\partial x_2 \partial x_1} = \frac{1}{64}$	$\frac{\partial^2\pi}{\partial x_2 \partial x_2} = -\frac{1}{96}$

e. By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

H =

$\frac{\partial^2\pi}{\partial x_1 \partial x_1} = \frac{-9}{64}$	$\frac{\partial^2\pi}{\partial x_1 \partial x_2} = \frac{1}{64}$
$\frac{\partial^2\pi}{\partial x_2 \partial x_1} = \frac{1}{64}$	$\frac{\partial^2\pi}{\partial x_2 \partial x_2} = -\frac{1}{96}$

The determinant of the Hessian is given by

$$
\begin{aligned}
\det[H] &= \left(-\frac{9}{64}\right)\left(-\frac{1}{96}\right) - \left(\left(\frac{1}{64}\right)\left(\frac{1}{64}\right)\right) \\
&= \left(-\frac{9}{2^6}\right)\left(-\frac{1}{2^5 \times 3}\right) - \left(\left(\frac{1}{2^6}\right)\left(\frac{1}{2^6}\right)\right) \\
&= \left(\frac{9}{2^6 \times 2^5 \times 3}\right) - \left(\frac{1}{2^{12}}\right) \\
&= \left(\frac{9 \times 2}{2^6 \times 2^6 \times 3}\right) - \left(\frac{3}{2^{12} \times 3}\right) \\
&= \left(\frac{18}{2^{12} \times 3}\right) - \left(\frac{3}{2^{12} \times 3}\right) \\
&= \frac{15}{2^{12} \times 3} \\
&= \frac{5}{2^{12}} = \frac{5}{4096}
\end{aligned}
$$

Given that the diagonals of the Hessian are negative and the determinant is postive, we know that the values of x we found represent a maximum.

Problem 4 (15 Points)**.** Find the listed partial derivatives of each of the following function.

$\mathcal{L}(x_1, x_2, \lambda) = \left(10x_1 + 30x_2 - x_1^2 + x_1x_2 - 2x_2^2\right) - \lambda\,(9x_1 + 6x_2 - 126)$

$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 10 - 2x_1 + x_2 - 9\lambda$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = 30 + x_1 - 4x_2 - 6\lambda$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = -\,(9x_1 + 6x_2 - 126)$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -2$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = 1$	-$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = 9$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = 1$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -4$	- $\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = 6$
- $\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = 9$	-$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = 6$	-$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0$