ECONOMICS 207
SPRING 2008
LABORATORY EXERCISE 10
KEY

For this laboratory exercise, consider the following matrices and vectors. You may want to tear this page off so it is easy to view.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 3 & -1 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 & -2 \\ 3 & -5 & -5 \\ -2 & 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} -5 & 2 & 0 \\ 1 & -1 & -1 \\ -4 & 2 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & -2 & 1 \\ -3 & 5 & -2 \\ 4 & -6 & 3 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & -4 \\ -4 & -5 & 3 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 1 & 1 \\ 7 & 11 & 10 \\ -8 & -13 & -12 \end{bmatrix}$$

$$a = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Date: August 5, 2008.

Problem 1.

a. Use elementary row operations to solve the following system of equations.

$$\begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \end{pmatrix}$$

First create the augmented matrix.

$$\tilde{A} = \begin{pmatrix} 2 & 3 & 8 \\ 3 & 1 & 5 \end{pmatrix} \tag{1}$$

Based on the augmented matrix, subtract the first row multiplied by 3/2 from the second row.

$$\begin{pmatrix} 2 & 3 & 8 \\ 3 - 2 \times 3/2 & 1 - 3 \times 3/2 & 5 - 8 \times 3/2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 8 \\ 0 & -7/2 & -7 \end{pmatrix} \tag{2}$$

Base on the matrix of the right side of equation (2), multiply the second row by $-2/7$.

$$\begin{pmatrix} 2 & 3 & 8 \\ 0 & -7/2 \times (-2/7) & -7 \times (-2/7) \end{pmatrix} = \begin{pmatrix} 2 & 3 & 8 \\ 0 & 1 & 2 \end{pmatrix} \tag{3}$$

Based on the matrix of the right side of equation (3), add the second row multiplied by -3 to the first row.

$$\begin{pmatrix} 2 & 3 + 1 \times (-3) & 8 + 2 \times (-3) \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix} \tag{4}$$

Based on the matrix of the right side of equation (4), divide the first row by 2.

$$\begin{pmatrix} 2/2 & 0 & 2/2 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \tag{5}$$

By observing the matrix on the right side of equation (5), the solution is $x_1 = 1,\ x_2 = 2$.

b. Use elementary row operations to solve the following system of equations.

$$\begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 20 \\ 10 \end{pmatrix}$$

First create the augmented matrix.

$$\tilde{A} = \begin{pmatrix} 3 & 6 & 20 \\ 2 & 4 & 10 \end{pmatrix} \tag{6}$$

Based on the augmented matrix, subtract the first row multiplied by 2/3 from the second row.

$$\begin{pmatrix} 3 & 6 & 20 \\ 2 - 3 \times 2/3 & 4 - 6 \times 2/3 & 10 - 20 \times 2/3 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 20 \\ 0 & 0 & -10/3 \end{pmatrix} \tag{7}$$

Base on the matrix of the right side of equation (7), there is no solution for this equation.

Problem 2.

a. Use elementary row operations to solve the following system of equations. The answers are x_1 = -9, x_2 = -4, x_3 = 3.

$$Dx = b$$

$$\begin{bmatrix} 1 & -2 & 1 \\ -3 & 5 & -2 \\ 4 & -6 & 3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

First create the augmented matrix $\tilde{A} = \begin{pmatrix} D & b \end{pmatrix}$.

$$\tilde{A} = \begin{pmatrix} 1 & -2 & 1 & 2 \\ -3 & 5 & -2 & 1 \\ 4 & -6 & 3 & -3 \end{pmatrix}$$

Based on the augmented matrix, add the first row multiplied by 3 to the second row.

$$\begin{pmatrix} 1 & -2 & 1 & 2 \\ -3+1\times 3 & 5-2\times 3 & -2+1\times 3 & 1+2\times 3 \\ 4 & -6 & 3 & -3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 & 2 \\ 0 & -1 & 1 & 7 \\ 4 & -6 & 3 & -3 \end{pmatrix} \tag{8}$$

Based on the matrix on the right side of equation (8), add the first row multiplied by -4 to the third row.

$$\begin{pmatrix} 1 & -2 & 1 & 2 \\ 0 & -1 & 1 & 7 \\ 4+1\times(-4) & -6-2\times(-4) & 3+1\times(-4) & -3+2\times(-4) \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 & 2 \\ 0 & -1 & 1 & 7 \\ 0 & 2 & -1 & -11 \end{pmatrix} \tag{9}$$

Based on the matrix on the right side of equation (9), add the second row multiplied by 2 to the third row.

$$\begin{pmatrix} 1 & -2 & 1 & 2 \\ 0 & -1 & 1 & 7 \\ 0 & 2+(-1)\times 2 & -1+1\times 2 & -11+7\times 2 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 & 2 \\ 0 & -1 & 1 & 7 \\ 0 & 0 & 1 & 3 \end{pmatrix} \tag{10}$$

Based on the matrix on the right side of equation (10), multiply the second row by -1.

$$\begin{pmatrix} 1 & -2 & 1 & 2 \\ 0 & -1\times(-1) & 1\times(-1) & 7\times(-1) \\ 0 & 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 & 2 \\ 0 & 1 & -1 & -7 \\ 0 & 0 & 1 & 3 \end{pmatrix} \tag{11}$$

Based on the matrix on the right side of equation (11), add the second row multiplied by 2 to the first row.

$$\begin{pmatrix} 1 & -2+2 & 1+(-1)\times 2 & 2+(-7)\times 2 \\ 0 & 1 & -1 & -7 \\ 0 & 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & -12 \\ 0 & 1 & -1 & -7 \\ 0 & 0 & 1 & 3 \end{pmatrix} \tag{12}$$

Based on the matrix on the right side of equation (12), add the third row to the first row.

$$\begin{pmatrix} 1 & 0 & -1+1 & -12+3 \\ 0 & 1 & -1 & -7 \\ 0 & 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -9 \\ 0 & 1 & -1 & -7 \\ 0 & 0 & 1 & 3 \end{pmatrix} \tag{13}$$

Based on the matrix on the right side of equation (13), add the third row to the second row.

$$\begin{pmatrix} 1 & 0 & 0 & -9 \\ 0 & 1 & -1+1 & -7+3 \\ 0 & 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{pmatrix} \tag{14}$$

By observing the matrix on the right side of equation (14), the solution is $x_1 = -9,\ x_2 = -4,\ x_3 = 3$.

b. Use elementary row operations to solve the following system of equations. Use elementary row operations to solve the following system of equations. The answers are $x_1 = 7$, $x_2 = -5$, $x_3 = 0$.

$$Ex = b$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & -4 \\ -4 & -5 & 3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

First create the augmented matrix $\tilde{A} = \begin{pmatrix} E & b \end{pmatrix}$.

$$\tilde{A} = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 3 & 4 & -4 & 1 \\ -4 & -5 & 3 & -3 \end{pmatrix}$$

Based on the augmented matrix, add the first row multiplied by -3 to the second row.

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 3+1\times(-3) & 4+1\times(-3) & -4+2\times(-3) & 1+2\times(-3) \\ -4 & -5 & 3 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & -10 & -5 \\ -4 & -5 & 3 & -3 \end{pmatrix} \tag{15}$$

Based on the matrix on the right side of equation (15), add the first row multiplied by 4 to the third row.

$$\begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & -10 & -5 \\ -4+1\times 4 & -5+1\times 4 & 3+2\times 4 & -3+2\times 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & -10 & -5 \\ 0 & -1 & 11 & 5 \end{pmatrix} \tag{16}$$

Based on the matrix on the right side of equation (16), add the third row to the first row.

$$\begin{pmatrix} 1 & 1-1 & 2+11 & 2+5 \\ 0 & 1 & -10 & -5 \\ 0 & -1 & 11 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 13 & 7 \\ 0 & 1 & -10 & -5 \\ 0 & -1 & 11 & 5 \end{pmatrix} \tag{17}$$

Based on the matrix on the right side of equation (17), add the second row to the third row.

$$\begin{pmatrix} 1 & 0 & 13 & 7 \\ 0 & 1 & -10 & -5 \\ 0 & -1+1 & 11-10 & 5-5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 13 & 7 \\ 0 & 1 & -10 & -5 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{18}$$

Based on the matrix on the right side of equation (18), add the third row multiplied by 10 to the second row.

$$\begin{pmatrix} 1 & 0 & 13 & 7 \\ 0 & 1 & -10+10 & -5 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 13 & 7 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{19}$$

Based on the matrix on the right side of equation (19), add the third row multiplied by −13 to the first row.

$$\begin{pmatrix} 1 & 0 & 13 - 1 \times 13 & 7 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{20}$$

By observing the matrix on the right side of equation (20), the solution is $x_1 = 7,\ x_2 = -5,\ x_3 = 0$.

Problem 3.

a. Use elementary row operations to find the inverse of the following matrix.

$$\begin{pmatrix} -2 & 3 \\ 3 & 4 \end{pmatrix}$$

The answer is $\begin{pmatrix} \frac{-4}{17} & \frac{3}{17} \\ \frac{3}{17} & \frac{2}{17} \end{pmatrix}$.

First, augment matrix $\begin{pmatrix} -2 & 3 \\ 3 & 4 \end{pmatrix}$ with an identity matrix. That is, let

$$\tilde{A} = \begin{pmatrix} -2 & 3 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{pmatrix} \tag{21}$$

Then using elementary row operations on matrix $\tilde{A}$ to create an identity matrix on the left side of the augmented matrix where the diagonal from left to right are all ones and all other entries are zeros.

Based on the augmented matrix, add the first row multiplied by $\frac{3}{2}$ to the second row.

$$\begin{pmatrix} -2 & 3 & 1 & 0 \\ 3 + (-2) \times \frac{3}{2} & 4 + 3 \times \frac{3}{2} & 0 + 1 \times \frac{3}{2} & 1 \end{pmatrix} = \begin{pmatrix} -2 & 3 & 1 & 0 \\ 0 & \frac{17}{2} & \frac{3}{2} & 1 \end{pmatrix} \tag{22}$$

Based on the matrix on the right side of equation (22), add the second row multiplied by $-\frac{6}{17}$ to the first row.

$$\begin{pmatrix} -2 & 3 + \frac{17}{2} \times \left(-\frac{6}{17}\right) & 1 + \frac{3}{2} \times \left(-\frac{6}{17}\right) & 0 + 1 \times \left(-\frac{6}{17}\right) \\ 0 & \frac{17}{2} & \frac{3}{2} & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 & \frac{8}{17} & -\frac{6}{17} \\ 0 & \frac{17}{2} & \frac{3}{2} & 1 \end{pmatrix} \tag{23}$$

Based on the matrix on the right side of equation (23), multiply the first row by $-\frac{1}{2}$ and multiply the second row by $\frac{2}{17}$

$$\begin{pmatrix} -2 \times \left(-\frac{1}{2}\right) & 0 & \frac{8}{17} \times \left(-\frac{1}{2}\right) & -\frac{6}{17} \times \left(-\frac{1}{2}\right) \\ 0 & \frac{17}{2} \times \frac{2}{17} & \frac{3}{2} \times \frac{2}{17} & 1 \times \frac{2}{17} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{4}{17} & \frac{3}{17} \\ 0 & 1 & \frac{3}{17} & \frac{2}{17} \end{pmatrix} \tag{24}$$

So the inverse of $\begin{pmatrix} -2 & 3 \\ 3 & 4 \end{pmatrix}$ is

$$\begin{pmatrix} -\frac{4}{17} & \frac{3}{17} \\ \frac{3}{17} & \frac{2}{17} \end{pmatrix}$$

b. Use elementary row operations to find the inverse of the following matrix.

$$\begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}$$

The answer is $\begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix}$.

First, augment matrix $\begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}$ with an identity matrix. That is, let

$$\tilde{A} = \begin{pmatrix} 5 & 2 & 1 & 0 \\ 7 & 3 & 0 & 1 \end{pmatrix} \tag{25}$$

Then using elementary row operations on matrix $\tilde{A}$ to create an identity matrix on the left side of the augmented matrix where the diagonal from left to right are all ones and all other entries are zeros.

Based on the augmented matrix $\tilde{A}$, add the first row multiplied by $-\frac{7}{5}$ to the second row.

$$\begin{pmatrix} 5 & 2 & 1 & 0 \\ 7 + 5 \times \left(-\frac{7}{5}\right) & 3 + 2 \times \left(-\frac{7}{5}\right) & 0 + 1 \times \left(-\frac{7}{5}\right) & 1 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 1 & 0 \\ 0 & \frac{1}{5} & -\frac{7}{5} & 1 \end{pmatrix} \tag{26}$$

Based on the matrix on the right side of equation (26), multiply the second row by 5.

$$\begin{pmatrix} 5 & 2 & 1 & 0 \\ 0 & \frac{1}{5} \times 5 & -\frac{7}{5} \times 5 & 1 \times 5 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 1 & 0 \\ 0 & 1 & -7 & 5 \end{pmatrix} \tag{27}$$

Based on the matrix on the right side of equation (27), add the second row multiplied by -2 to the first row.

$$\begin{pmatrix} 5 & 2 + 1 \times (-2) & 1 + (-7) \times (-2) & 0 + 5 \times (-2) \\ 0 & 1 & -7 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 15 & -10 \\ 0 & 1 & -7 & 5 \end{pmatrix} \tag{28}$$

Based on the matrix on the right side of equation (28), divide the first row by 5.

$$\begin{pmatrix} 5/5 & 0 & 15/5 & -10/5 \\ 0 & 1 & -7 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -7 & 5 \end{pmatrix}$$

So the inverse of $\begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}$ is

$$\begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix}$$

Problem 4.

a. Use elementary row operations to find the inverse of the following matrix.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 3 & -1 & 6 \end{bmatrix}$$

The answer is

$$A^{-1} = \begin{bmatrix} 14 & -5 & -1 \\ -24 & 9 & 2 \\ -11 & 4 & 1 \end{bmatrix}$$

First, augment matrix $\begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 3 & -1 & 6 \end{pmatrix}$ with an identity matrix. That is, let

$$\tilde{A} = \begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & 3 & -4 & 0 & 1 & 0 \\ 3 & -1 & 6 & 0 & 0 & 1 \end{pmatrix} \tag{29}$$

Then using elementary row operations on matrix $\tilde{A}$ to create an identity matrix on the left side of the augmented matrix where the diagonal from left to right are all ones and all other entries are zeros.

Based on the augmented matrix $\tilde{A}$, add the first row multiplied by -2 to the second row.

$$\begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 2+1\times(-2) & 3+1\times(-2) & -4+(-1)\times(-2) & 0+1\times(-2) & 1 & 0 \\ 3 & -1 & 6 & 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 3 & -1 & 6 & 0 & 0 & 1 \end{pmatrix} \tag{30}$$

Based on the matrix on the right side of equation (30), add the first row multiplied by -3 to the third row.

$$\begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 3+1\times(-3) & -1+1\times(-3) & 6+(-1)\times(-3) & 0+1\times(-3) & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & -4 & 9 & -3 & 0 & 1 \end{pmatrix} \tag{31}$$

Based on the matrix on the right side of equation (31), add the second row multiplied by 4 to the third row.

$$\begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & -4+1\times 4 & 9+(-2)\times 4 & -3+(-2)\times 4 & 0+1\times 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -11 & 4 & 1 \end{pmatrix} \tag{32}$$

Based on the matrix on the right side of equation (32), subtract the second row from the first row.

$$\begin{pmatrix} 1 & 1-1 & -1-(-2) & 1-(-2) & 0-1 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -11 & 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -11 & 4 & 1 \end{pmatrix} \tag{33}$$

Based on the matrix on the right side of equation (33), subtract the third row from the first row.

$$\begin{pmatrix} 1 & 0 & 1-1 & 3-(-11) & -1-4 & 0-1 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -11 & 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 14 & -5 & -1 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -11 & 4 & 1 \end{pmatrix} \tag{34}$$

Based on the matrix on the right side of equation (34), add the third row multiplied by 2 to the second row.

$$\begin{pmatrix} 1 & 0 & 0 & 14 & -5 & -1 \\ 0 & 1 & -2+1\times 2 & -2+(-11)\times 2 & 1+4\times 2 & 0+1\times 2 \\ 0 & 0 & 1 & -11 & 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 14 & -5 & -1 \\ 0 & 1 & 0 & -24 & 9 & 2 \\ 0 & 0 & 1 & -11 & 4 & 1 \end{pmatrix} \tag{35}$$

So the inverse of $\begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 3 & -1 & 6 \end{pmatrix}$ is

$$\begin{pmatrix} 14 & -5 & -1 \\ -24 & 9 & 2 \\ -11 & 4 & 1 \end{pmatrix}$$

b. Use elementary row operations to find the inverse of the following matrix.

$$F = \begin{bmatrix} 1 & 1 & 1 \\ 7 & 11 & 10 \\ -8 & -13 & -12 \end{bmatrix}$$

The answer is

$$F^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ -4 & 4 & 3 \\ 3 & -5 & -4 \end{bmatrix}$$

First, augment matrix $\begin{pmatrix} 1 & 1 & 1 \\ 7 & 11 & 10 \\ -8 & -13 & -12 \end{pmatrix}$ with an identity matrix. That is, let

$$\tilde{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 7 & 11 & 10 & 0 & 1 & 0 \\ -8 & -13 & -12 & 0 & 0 & 1 \end{pmatrix} \tag{36}$$

Then using elementary row operations on matrix $\tilde{A}$ to create an identity matrix on the left side of the augmented matrix where the diagonal from left to right are all ones and all other entries are zeros.

Based on the augmented matrix $\tilde{A}$, add the first row multiplied by -7 to the second row.

$$\begin{aligned} &\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 7+1\times(-7) & 11+1\times(-7) & 10+1\times(-7) & 0+1\times(-7) & 1 & 0 \\ -8 & -13 & -12 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 4 & 3 & -7 & 1 & 0 \\ -8 & -13 & -12 & 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{37}$$

Based on the matrix on the right side of equation (37), add the first row multiplied by 8 to the third row.

$$\begin{aligned} &\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 4 & 3 & -7 & 1 & 0 \\ -8+1 & -13+1 & -12+1 & 0+1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 4 & 3 & -7 & 1 & 0 \\ 0 & -5 & -4 & 8 & 0 & 1 \end{pmatrix} \end{aligned} \tag{38}$$

Based on the matrix on the right side of equation (38), add the second row multiplied by 5/4 to the third row.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 4 & 3 & -7 & 1 & 0 \\ 0 & -5+4\times\frac{5}{4} & -4+3\times\frac{5}{4} & 8+(-7)\times\frac{5}{4} & 0+1\times\frac{5}{4} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 4 & 3 & -7 & 1 & 0 \\ 0 & 0 & -\frac{1}{4} & -\frac{3}{4} & \frac{5}{4} & 1 \end{pmatrix} \tag{39}$$

Based on the matrix on the right side of equation (39), multiply the third row by −4.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 4 & 3 & -7 & 1 & 0 \\ 0 & 0 & -\frac{1}{4}\times(-4) & -\frac{3}{4}\times(-4) & \frac{5}{4}\times(-4) & 1\times(-4) \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 4 & 3 & -7 & 1 & 0 \\ 0 & 0 & 1 & 3 & -5 & -4 \end{pmatrix} \tag{40}$$

Based on the matrix on the right side of equation (40), subtract the third row multiplied by 3 from the second row.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 4 & 3-1\times 3 & -7-3\times 3 & 1-(-5)\times 3 & 0-(-4)\times 3 \\ 0 & 0 & 1 & 3 & -5 & -4 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 4 & 0 & -16 & 16 & 12 \\ 0 & 0 & 1 & 3 & -5 & -4 \end{pmatrix} \tag{41}$$

Based on the matrix on the right side of equation (41), divide the second row by 4.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 4/4 & 0 & -16/4 & 16/4 & 12/4 \\ 0 & 0 & 1 & 3 & -5 & -4 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 4 & 3 \\ 0 & 0 & 1 & 3 & -5 & -4 \end{pmatrix} \tag{42}$$

Based on the matrix on the right side of equation (42), subtract the second row from the first row. divide the second row by 4.

$$\begin{pmatrix} 1 & 1-1 & 1 & 1-(-4) & 0-4 & 0-3 \\ 0 & 1 & 0 & -4 & 4 & 3 \\ 0 & 0 & 1 & 3 & -5 & -4 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 1 & 5 & -4 & -3 \\ 0 & 1 & 0 & -4 & 4 & 3 \\ 0 & 0 & 1 & 3 & -5 & -4 \end{pmatrix} \tag{43}$$

Based on the matrix on the right side of equation (43), subtract the third row from the first row. divide the second row by 4.

$$\begin{pmatrix} 1 & 0 & 1-1 & 5-3 & -4-(-5) & -3-(-4) \\ 0 & 1 & 0 & -4 & 4 & 3 \\ 0 & 0 & 1 & 3 & -5 & -4 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 2 & 1 & 1 \\ 0 & 1 & 0 & -4 & 4 & 3 \\ 0 & 0 & 1 & 3 & -5 & -4 \end{pmatrix} \tag{44}$$

So the inverse of $\begin{pmatrix} 1 & 1 & 1 \\ 7 & 11 & 10 \\ -8 & -13 & -12 \end{pmatrix}$ is

$$\begin{pmatrix} 2 & 1 & 1 \\ -4 & 4 & 3 \\ 3 & -5 & -4 \end{pmatrix}$$

Problem 5.

a. Use the inverse you found in problem 3a to solve the following equation.

$$\begin{pmatrix} -2 & 3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 11 \\ 9 \end{pmatrix}$$

The answer is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

Using the inverse of matrix $\begin{pmatrix} -2 & 3 \\ 3 & 4 \end{pmatrix}$, the solution is given by

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} -2 & 3 \\ 3 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 11 \\ 9 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{4}{17} & \frac{3}{17} \\ \frac{3}{17} & \frac{2}{17} \end{pmatrix} \begin{pmatrix} 11 \\ 9 \end{pmatrix} \\ &= \begin{pmatrix} \frac{-4}{17} \times 11 + \frac{3}{17} \times 9 \\ \frac{3}{17} \times 11 + \frac{2}{17} \times 9 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 3 \end{pmatrix} \end{aligned}$$

b. Use the inverse you found in problem 3b to solve the following equation.

$$\begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

Using the inverse of matrix $\begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}$, the solution is given by

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 3 & -2 \\ -7 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 3 \times 3 - 2 \times 5 \\ -7 \times 3 + 5 \times 5 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 4 \end{pmatrix} \end{aligned}$$

Problem 6. Use the inverse you found in problem 4a to solve the following equation.

$$Ax = a$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 3 & -1 & 6 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Using the inverse of matrix $\begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 3 & -1 & 6 \end{pmatrix}$, the solution is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 3 & -4 \\ 3 & -1 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & -5 & -1 \\ -24 & 9 & 2 \\ -11 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 14 \\ -24 \\ -11 \end{pmatrix}$$

Problem 7. Compute the determinants of the following matrices.

a.
$$\begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = 2 \times 1 - 3 \times 3 = -7$$

b.
$$\begin{pmatrix} -2 & 3 \\ 3 & 4 \end{pmatrix}$$

$$\begin{vmatrix} -2 & 3 \\ 3 & 4 \end{vmatrix} = -2 \times 4 - 3 \times 3 = -17$$

c.
$$\begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix}$$

$$\begin{vmatrix} 3 & 6 \\ 2 & 4 \end{vmatrix} = 3 \times 4 - 6 \times 2 = 0$$

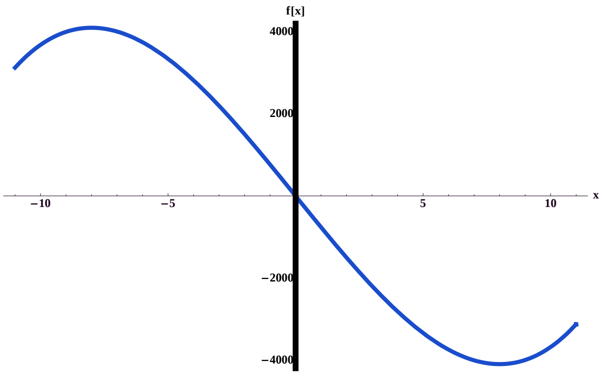
d.
$$\begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix}$$

$$\begin{vmatrix} 5 & 2 \\ 7 & 3 \end{vmatrix} = 5 \times 3 - 2 \times 7 = 1$$

Problem 8. For each of the following problems, find the critical points. For each critical point state whether the function is at a relative maximum, relative minimum, or otherwise. Check to see if there are potential points of inflection **at points other than** critical points.

a. $f(x) = 4x^3 - 768x$.

FIGURE 1. $f(x) = 4x^3 - 768x$



The derivatives of $f(x) = 4x^3 - 768x$ are given by

$$f'(x) = 12x^2 - 768 \tag{45}$$

$$f''(x) = 24x \tag{46}$$

$$f^{(3)}(x) = 24 \tag{47}$$

Set the first derivative, equation (45), to zero.

$$\begin{aligned} f'(x) = 12x^2 - 768 &= 0 \\ \Rightarrow \qquad x^2 &= 768/12 = 64 \\ \Rightarrow \qquad x &\pm 8 \end{aligned}$$

When $x = \pm 8$, the second derivative, equation (46), is respectively given by

$$f''(8) = 24 \times 8 > 0$$

$$f''(-8) = 24 \times (-8) < 0$$

So $x = 8$ is relative minimum point, and $x = -8$ is relative maximum point.

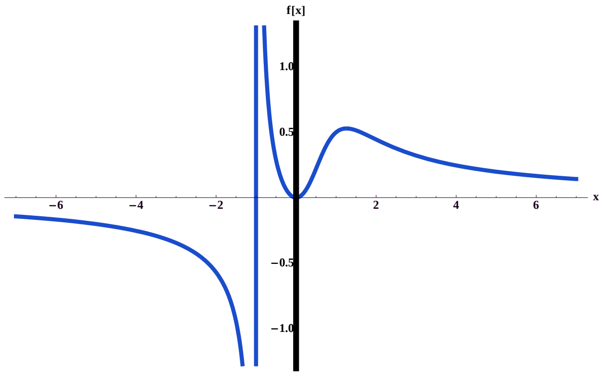
Set the second derivative, equation (46), to zero.

$$\begin{aligned} f''(x) = 24x &= 0 \\ \Rightarrow \qquad x &= 0 \end{aligned}$$

When $x = 0$, the third derivative, equation (47), is 24, which is not equal to zero. As a result, $x = 0$ is an inflection point.

b. $f(x) = \frac{x^2}{x^3+1}$

Figure 2. $f(x) = \frac{x^2}{x^3+1}$



Hints: The second derivative of the function is $\frac{2(x^6-7x^3+1)}{(x^3+1)^3}$. To solve the following equation $x^6 - 7x^3 + 1 = 0$, use the substitution z = x³. The real points of inflection (you still need to find them) are $\left(\frac{7\pm3\sqrt{5}}{2}\right)^{1/3}$.

The derivatives of $f(x) = \frac{x^2}{x^3+1}$ are given by

$$f'(x) = \frac{2x(x^3+1) - x^2(3x^2)}{(x^3+1)^2} = \frac{-x^4+2x}{(x^3+1)^2}$$

$$f''(x) = \frac{(-4x^3+2)(x^3+1)^2 - (-x^4+2x)[2(x^3+1)(3x^2)]}{(x^3+1)^4}$$

$$= \frac{(-4x^3+2)(x^3+1) - 2(-x^4+2x)(3x^2)}{(x^3+1)^3}$$

$$= \frac{-4x^6+2x^3-4x^3+2+6x^6-12x^3}{(x^3+1)^3}$$

$$= \frac{2(x^6-7x^3+1)}{(x^3+1)^3}$$

Consider $x^3 = z$ in the second derivative and using chain rule, the third derivative is then the derivative with respect to z multiplied by $\frac{dz}{dz} = 3x^2$. That is,

$$f^{(3)}(x) = 2\frac{(2x^3-7)(x^3+1)^3 - (x^6-7x^3+1)[3(x^3+1)^2]}{(x^3+1)^6} \cdot (3x^2)$$

$$= 6x^2\frac{(2x^3-7)(x^3+1) - 3(x^6-7x^3+1)}{(x^3+1)^4}$$

$$= \frac{-6x^2(x^6-16x^3+10)}{(x^3+1)^4}$$

Set the first derivative, $\dfrac{-x^4+2x}{(x^3+1)^2}=0$.

$$\begin{aligned} \frac{-x^4+2x}{(x^3+1)^2} &= 0 \\ \Rightarrow \quad x(x^3-2) &= 0 \\ \Rightarrow \quad x=0 \ \text{ or } \ x &= \sqrt[3]{2} \end{aligned}$$

Check the second derivative for $x=0$ and $x=\sqrt[3]{2}$.

$$f''(0)=\frac{2(0^6-70^3+1)}{(0^3+1)^3}=2>0$$

$$f''(\sqrt[3]{2})=\frac{2(2^2-7\times 2+1)}{(2+1)^3}=-\frac{2}{3}<0$$

So $x=0$ is a relative minimum point, and $x=\sqrt[3]{2}$ is a relative maximum point.

Set the second derivative, $\dfrac{2(x^6-7x^3+1)}{(x^3+1)^3}$, to zero.

$$\begin{aligned} \frac{2(x^6-7x^3+1)}{(x^3+1)^3} &= 0 \\ \Rightarrow \quad x^6-7x^3+1=0 \ \text{ and } \ x^3 &\neq -1 \\ \Rightarrow \quad (x^3)^2-7x^3+1=0 \ \text{ and } \ x^3 &\neq -1 \\ \Rightarrow \quad x^3 &= \tfrac{7\pm\sqrt{(-7)^2-4}}{2}=\tfrac{7\pm 3\sqrt{5}}{2} \\ \Rightarrow \quad x &= \left(\tfrac{7\pm 3\sqrt{5}}{2}\right)^{1/3} \end{aligned}$$

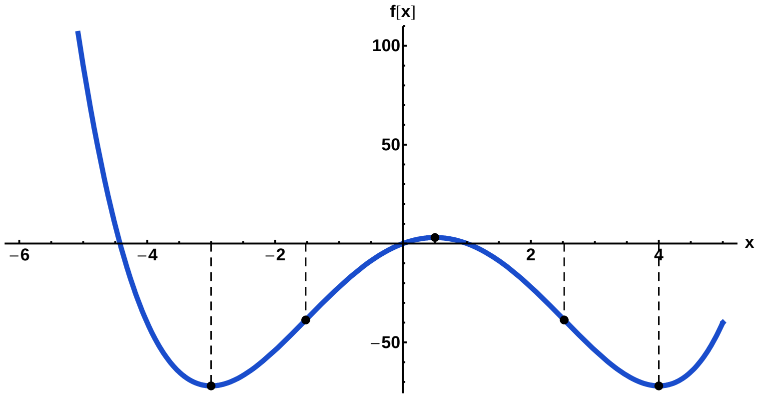
So $x=\left(\frac{7\pm 3\sqrt{5}}{2}\right)^{1/3}$ are two potential inflection points. We still need to check the third derivative. When $x=\left(\frac{7\pm 3\sqrt{5}}{2}\right)^{1/3}$, that is $x^6-7x^3+1=0$,

$$\begin{aligned} f^{(3)}(x) &= \frac{-6x^2(x^6-16x^3+10)}{(x^3+1)^4} \\ &= \frac{-6x^2(x^6-7x^3+1-9x^3+9)}{(x^3+1)^4} \\ &= \frac{-6x^2(-9x^3+9)}{(x^3+1)^4} \neq 0 \end{aligned}$$

So $x=\left(\frac{7\pm 3\sqrt{5}}{2}\right)^{1/3}$ are two inflection points.

c. $f(x) \;=\; \dfrac{x^4}{2} - x^3 - \frac{23}{2}x^2 + 12x$

FIGURE 3. $f(x) \;=\; \dfrac{x^4}{2} - x^3 - \frac{23}{2}x^2 + 12x$



The inflection points are $\frac{3\pm 7\sqrt{3}}{6}$.

The derivatives of $f(x) \;=\; \dfrac{x^4}{2} - x^3 - \frac{23}{2}x^2 + 12x$ are given by

$$f'(x) = 2x^3 - 3x^2 - 23x + 12 \tag{48}$$

$$f''(x) = 6x^2 - 6x - 23 \tag{49}$$

$$f^{(3)}(x) = 12x - 6 \tag{50}$$

Set the first derivative, equation (48), to be zero.

$$f'(x) = 2x^3 - 3x^2 - 23x + 12 = 0 \tag{51}$$

By guess, $x = 4$ is a solution of equation (51). Then we try to factor $2x^3 - 3x^2 - 23x + 12$ by $x - 4$. That is,

$$\begin{array}{r|rrrr}
 & 2x^2 & +5x & -3 & \\
\hline
x-4 & 2x^3 - 3x^2 & - 23x & + 12 & \\
 & -2x^3 + 8x^2 & & & \\
\hline
 & 5x^2 & - 23x & & \\
 & -5x^2 & + 20x & & \\
\hline
 & & -3x & + 12 & \\
 & & 3x & - 12 & \\
\hline
 & & & 0 &
\end{array}$$

As a result,

$$\begin{aligned}
2x^3 - 3x^2 - 23x + 12 &= 0 \\
\Rightarrow \quad (x-4)(2x^2 + 5x - 3) &= 0 \\
\Rightarrow \quad (x-4)(2x-1)(x+3) &= 0 \\
\Rightarrow \quad x = 4 \quad \text{or} \quad x = 1/2 \quad & \text{or} \quad x = -3
\end{aligned}$$

Check the second derivatives for $x = 4,\ x = 1/2,$ and $x = -3$.

$$f''(4) = 6 \times 4^2 - 6 \times 4 - 23 = 49 > 0$$

$$f''(1/2) = 6 \times (1/2)^2 - 6 \times (1/2) - 23 = -24.5 < 0$$

$$f''(-3) = 6 \times (-3)^2 - 6 \times (-3) - 23 = 49 > 0$$

In addition, $f(4) = f(-3) = -72$. So $x = 1/2$ is a relative maximum point. $x = -3$ and $x = 4$ are global minimum points.

Set the second derivative, equation (49), to zero.

$$\begin{aligned} f''(x) = 6x^2 - 6x - 23 &= 0 \\ \Rightarrow \qquad x &= \tfrac{6 \pm \sqrt{36 - 4 \times 6 \times (-23)}}{12} \\ \Rightarrow \qquad x &= \tfrac{3 \pm 7\sqrt{3}}{6} \end{aligned}$$

Since the third derivative, equation (50), is zero only when $x = 1/2$, $x = \frac{3 \pm 7\sqrt{3}}{6}$ are inflection points.