ECONOMICS 207
SPRING 2008
LABORATORY EXERCISE 11
KEY

Problem 1. Consider the following matrix and vector.

$$P = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}, \quad p = \begin{bmatrix} 8 \\ 18 \end{bmatrix},$$

a. Use elementary row operations to find both the inverse of P and solve the equation Px=p in one set of operations.

First, augment matrix P with an 2×2 identity matrix and p. That is,

$$\begin{aligned} \tilde{A} &= \begin{pmatrix} P & I_{2\times 2} & p \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 1 & 0 & 8 \\ 2 & 5 & 0 & 1 & 18 \end{pmatrix} \end{aligned}$$

Then using elementary row operation on matrix $\tilde{A}$ to create an identity matrix on the left side.

Based on the augmented matrix $\tilde{A}$, subtract the first row multiplied by 2 from the second row.

$$\begin{pmatrix} 1 & 2 & 1 & 0 & 8 \\ 2-1\times 2 & 5-2\times 2 & 0-1\times 2 & 1 & 18-8\times 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 0 & 8 \\ 0 & 1 & -2 & 1 & 2 \end{pmatrix} \tag{1}$$

Based on the matrix on the right side of equation (1), subtract the second row multiplied by 2 from the first row.

$$\begin{pmatrix} 1 & 2-1\times 2 & 1-(-2)\times 2 & 0-1\times 2 & 8-2\times 2 \\ 0 & 1 & -2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 5 & -2 & 4 \\ 0 & 1 & -2 & 1 & 2 \end{pmatrix} \tag{2}$$

So $P^{-1} = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$. And the solution for $P\mathbf{x} = p$ is $\mathbf{x} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

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b. Find the determinant of the matrix P.

$$P = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}, \quad p = \begin{bmatrix} 8 \\ 18 \end{bmatrix},$$

$$|P| = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 1 \times 5 - 2 \times 2 = 1$$

c. Find the inverse of the matrix P using the cofactor/adjoint method.

The adjoint of matrix P is given by

$$\text{adj}(P) = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}^T = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$$

Then the inverse of matrix P is given by

$$P^{-1} = \frac{\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}}{|P|} = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$$

d. Solve the equation Px=p using the inverse you found in part 1c

The solution is given by

$$\mathbf{x} = P^{-1}p = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 8 \\ 18 \end{pmatrix} = \begin{pmatrix} 5 \times 8 - 2 \times 18 \\ -2 \times 8 + 1 \times 18 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

e. Solve the equation Px=p using Cramer's rule.

$$P = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}, \quad p = \begin{bmatrix} 8 \\ 18 \end{bmatrix},$$

Using Cramer's rule, the solution is given by

$$\mathbf{x} = \begin{pmatrix} \dfrac{\begin{vmatrix} 8 & 2 \\ 18 & 5 \end{vmatrix}}{|P|} \\ \dfrac{\begin{vmatrix} 1 & 8 \\ 2 & 18 \end{vmatrix}}{|P|} \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

Problem 2. Consider the following matrix and vector.

$$Q = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

a. Use elementary row operations to find both the inverse of Q and solve the equation Qx=q in one set of operations.

First augment matrix Q with an 2×2 identity matrix and matrix q. That is,

$$\begin{aligned} \tilde{A} &= \begin{pmatrix} Q & I_{2\times 2} & q \end{pmatrix} \\ &= \begin{pmatrix} 2 & 4 & 1 & 0 & 0 \\ 1 & 3 & 0 & 1 & -1 \end{pmatrix} \end{aligned}$$

Then using elementary row operations on the augmented matrix $\tilde{A}$ to create an identity matrix on the left side.

Based on the augmented matrix $\tilde{A}$, subtract the first row multiplied by $1/2$ from the second row.

$$\begin{pmatrix} 2 & 4 & 1 & 0 & 0 \\ 1-2\times(1/2) & 3-4\times(1/2) & 0-1\times(1/2) & 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & 1 & -1 \end{pmatrix} \tag{3}$$

Based on the augmented matrix on the right side of equation (3), subtract the second row multiplied by 4 from the first row.

$$\begin{pmatrix} 2 & 4-1\times 4 & 1-(-1/2)\times 4 & 0-1\times(4) & 0-(-1)\times(4) \\ 0 & 1 & -1/2 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 & -4 & 4 \\ 0 & 1 & -1/2 & 1 & -1 \end{pmatrix} \tag{4}$$

Based on the augmented matrix on the right side of equation (4), divide the first row by 2.

$$\begin{pmatrix} 2/2 & 0 & 3/2 & -4/2 & 4/2 \\ 0 & 1 & -1/2 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3/2 & -2 & 2 \\ 0 & 1 & -1/2 & 1 & -1 \end{pmatrix} \tag{5}$$

So $Q^{-1} = \begin{pmatrix} 3/2 & -2 \\ -1/2 & 1 \end{pmatrix}$.

And the solution is given by $\mathbf{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

b. Find the determinant of the matrix Q.

$$Q = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

$$\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 2 \times 3 - 4 \times 1 = 2$$

c. Find the inverse of the matrix Q using the cofactor/adjoint method.

The adjoint of matrix Q is given by

$$\text{adj}(Q) = \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}^T = \begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix}$$

Then the inverse of matrix Q is given by

$$\begin{aligned} Q^{-1} &= \frac{\begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix}}{|Q|} \\ &= \begin{pmatrix} 3/2 & -2 \\ -1/2 & 1 \end{pmatrix} \end{aligned}$$

d. Solve the equation Qx=q using the inverse you found in part 2c

The solution is given by

$$\begin{aligned} \mathbf{x} &= Q^{-1}q \\ &= \begin{pmatrix} 3/2 & -2 \\ -1/2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} (3/2) \times 0 + (-2) \times (-1) \\ (-1/2) \times 0 + 1 \times (-1) \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ -1 \end{pmatrix} \end{aligned}$$

e. Solve the equation Qx=q using Cramer's rule.

$$Q = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Using Cramer's rule, the solution is given by

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} \dfrac{\begin{vmatrix} 0 & 4 \\ -1 & 3 \end{vmatrix}}{|Q|} \\ \dfrac{\begin{vmatrix} 2 & 0 \\ 1 & -1 \end{vmatrix}}{|Q|} \end{pmatrix} \\ &= \begin{pmatrix} 4/2 \\ -2/2 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ -1 \end{pmatrix} \end{aligned}$$

Problem 3. Consider the following matrix and vector.

$$F = \begin{bmatrix} 1 & -1 & 2 \\ -4 & 5 & -6 \\ 2 & -3 & 3 \end{bmatrix}, \quad f = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

a. Use elementary row operations to find both the inverse of F and solve the equation Fx=f in one set of operations.

First augment the matrix F with an identity matrix $I_{3\times 3}$ and matrix f. That is,

$$\begin{aligned} \tilde{A} &= \begin{pmatrix} F & I_{3\times 3} & f \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 & 2 & 1 & 0 & 0 & 3 \\ -4 & 5 & -6 & 0 & 1 & 0 & -2 \\ 2 & -3 & 3 & 0 & 0 & 1 & -1 \end{pmatrix} \end{aligned}$$

Then using elementary row operations on matrix $\tilde{A}$ to make the left side be an identity matrix. Based on $\tilde{A}$, add the first row multiplied by 4 to the second row.

$$\begin{pmatrix} 1 & -1 & 2 & 1 & 0 & 0 & 3 \\ -4+1\times 4 & 5+(-1)\times 4 & -6+2\times 4 & 0+1\times 4 & 1 & 0 & -2+3\times 4 \\ 2 & -3 & 3 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 & 2 & 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & 4 & 1 & 0 & 10 \\ 2 & -3 & 3 & 0 & 0 & 1 & -1 \end{pmatrix} \tag{6}$$

Based on the matrix on the right side of equation (6), subtract the first row multiplied by 2 from the third row.

$$\begin{pmatrix} 1 & -1 & 2 & 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & 4 & 1 & 0 & 10 \\ 2-1\times 2 & -3-(-1)\times 2 & 3-2\times 2 & 0-1\times 2 & 0 & 1 & -1-3\times 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 & 2 & 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & 4 & 1 & 0 & 10 \\ 0 & -1 & -1 & -2 & 0 & 1 & -7 \end{pmatrix} \tag{7}$$

Based on the matrix on the right side of equation (7), add the second row to the third row.

$$\begin{pmatrix} 1 & -1 & 2 & 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & 4 & 1 & 0 & 10 \\ 0 & -1+1 & -1+2 & -2+4 & 0+1 & 1 & -7+10 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 & 2 & 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & 4 & 1 & 0 & 10 \\ 0 & 0 & 1 & 2 & 1 & 1 & 3 \end{pmatrix} \tag{8}$$

Based on the matrix on the right side of equation (8), add the third row multiplied by -2 to the first and the second row.

$$\begin{pmatrix} 1 & -1 & 2+1\times(-2) & 1+2\times(-2) & 0+1\times(-2) & 0+1\times(-2) & 3+3\times(-2) \\ 0 & 1 & 2+1\times(-2) & 4+2\times(-2) & 1+1\times(-2) & 0+1\times(-2) & 10+3\times(-2) \\ 0 & 0 & 1 & 2 & 1 & 1 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -1 & 0 & -3 & -2 & -2 & -3 \\ 0 & 1 & 0 & 0 & -1 & -2 & 4 \\ 0 & 0 & 1 & 2 & 1 & 1 & 3 \end{pmatrix} \tag{9}$$

Based on the matrix on the right side of equation (9), add the second row to the first row.

$$\begin{pmatrix} 1 & -1+1 & 0 & -3 & -2-1 & -2-2 & -3+4 \\ 0 & 1 & 0 & 0 & -1 & -2 & 4 \\ 0 & 0 & 1 & 2 & 1 & 1 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & -3 & -3 & -4 & 1 \\ 0 & 1 & 0 & 0 & -1 & -2 & 4 \\ 0 & 0 & 1 & 2 & 1 & 1 & 3 \end{pmatrix} \tag{10}$$

So $F^{-1} = \begin{pmatrix} -3 & -3 & -4 \\ 0 & -1 & -2 \\ 2 & 1 & 1 \end{pmatrix}$. And the solution of $F\mathbf{x} = f$ is given by

$$\mathbf{x} = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$$

b. Find the determinant of the matrix F.

$$F = \begin{bmatrix} 1 & -1 & 2 \\ -4 & 5 & -6 \\ 2 & -3 & 3 \end{bmatrix}, \quad f = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$$\begin{vmatrix} 1 & -1 & 2 \\ -4 & 5 & -6 \\ 2 & -3 & 3 \end{vmatrix} = 1 \times \begin{vmatrix} 5 & -6 \\ -3 & 3 \end{vmatrix} - (-1) \times \begin{vmatrix} -4 & -6 \\ 2 & 3 \end{vmatrix} + 2 \times \begin{vmatrix} -4 & 5 \\ 2 & -3 \end{vmatrix}$$

$$= -3 + 0 + 4 = 1$$

c. Find the inverse of the matrix F using the cofactor/adjoint method.

$$F = \begin{bmatrix} 1 & -1 & 2 \\ -4 & 5 & -6 \\ 2 & -3 & 3 \end{bmatrix}, \quad f = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

The adjoint of matrix F is given by

$$\text{adj}(F) = \begin{pmatrix} \begin{vmatrix} 5 & -6 \\ -3 & 3 \end{vmatrix} & -\begin{vmatrix} -4 & -6 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} -4 & 5 \\ 2 & -3 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ -3 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix} \\ \begin{vmatrix} -1 & 2 \\ 5 & -6 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ -4 & -6 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ -4 & 5 \end{vmatrix} \end{pmatrix}^T$$

$$= \begin{pmatrix} -3 & 0 & 2 \\ -3 & -1 & 1 \\ -4 & -2 & 1 \end{pmatrix}^T$$

$$= \begin{pmatrix} -3 & -3 & -4 \\ 0 & -1 & -2 \\ 2 & 1 & 1 \end{pmatrix}$$

So $F^{-1} = \text{adj}(F)/|F| = \text{adj}(F)$. That is ,

$$F^{-1} = \begin{pmatrix} -3 & -3 & -4 \\ 0 & -1 & -2 \\ 2 & 1 & 1 \end{pmatrix}$$

d. Solve the equation Fx=f using the inverse you found in part 3c

$$F = \begin{bmatrix} 1 & -1 & 2 \\ -4 & 5 & -6 \\ 2 & -3 & 3 \end{bmatrix}, \quad f = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

The solution is given by

$$\begin{aligned} \mathbf{x} &= F^{-1}f \\ &= \begin{pmatrix} -3 & -3 & -4 \\ 0 & -1 & -2 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -3 \times 3 + (-3) \times (-2) + (-4) \times (-1) \\ 0 \times 3 + (-1) \times (-2) + (-2) \times (-1) \\ 2 \times 3 + 1 \times (-2) + 1 \times (-1) \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \end{aligned}$$

e. Solve the equation Fx=f using Cramer's rule.

Using Cramer's rule, the solution is given by

$$\mathbf{x} = \frac{\begin{pmatrix} \begin{vmatrix} 3 & -1 & 2 \\ -2 & 5 & -6 \\ -1 & -3 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 & 2 \\ -4 & -2 & -6 \\ 2 & -1 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 & 3 \\ -4 & 5 & -2 \\ 2 & -3 & -1 \end{vmatrix} \end{pmatrix}}{|F|}$$

$$= \begin{pmatrix} 3 \times \begin{vmatrix} 5 & -6 \\ -3 & 3 \end{vmatrix} - (-1) \times \begin{vmatrix} -2 & -6 \\ -1 & 3 \end{vmatrix} + 2 \times \begin{vmatrix} -2 & 5 \\ -1 & -3 \end{vmatrix} \\ 1 \times \begin{vmatrix} -2 & -6 \\ -1 & 3 \end{vmatrix} - 3 \times \begin{vmatrix} -4 & -6 \\ 2 & 3 \end{vmatrix} + 2 \times \begin{vmatrix} -4 & -2 \\ 2 & -1 \end{vmatrix} \\ 1 \times \begin{vmatrix} 5 & -2 \\ -3 & -1 \end{vmatrix} - (-1) \times \begin{vmatrix} -4 & -2 \\ 2 & -1 \end{vmatrix} + 3 \times \begin{vmatrix} -4 & 5 \\ 2 & -3 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 3 \times (-3) + (-12) + 2 \times 11 \\ 1 \times (-12) - 3 \times 0 + 2 \times 8 \\ 1 \times (-11) + 1 \times 8 + 3 \times 2 \end{pmatrix}$$

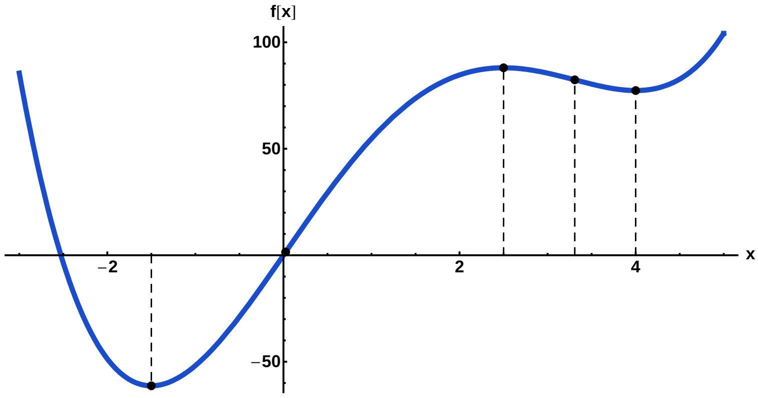
$$= \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$$

Problem 4. This is a free problem.

Problem 5. For each of the following problems, find the critical points. For each critical point state whether the function is at a relative maximum, relative minimum, or otherwise. Check to see if there are potential points of inflection **at points other than** critical points.

a. $f(x) = x^4 - \frac{20}{3}x^3 + \frac{x^2}{2} + 60x$.

FIGURE 1. $f(x) = x^4 - \frac{20}{3}x^3 + \frac{x^2}{2} + 60x$



The inflection points are $\frac{1}{6}\left(10 \pm \sqrt{97}\right)$.

The derivatives of $f(x) = x^4 - \frac{20}{3}x^3 + \frac{x^2}{2} + 60x$ are given by

$$f'(x) = 4x^3 - 20x^2 + x + 60 \tag{11}$$

$$f''(x) = 12x^2 - 40x + 1 \tag{12}$$

$$f^{(3)} = 24x - 40 \tag{13}$$

Set the first derivative, equation (11), to be zero. That is,

$$f'(x) = 4x^3 - 20x^2 + x + 60 = 0 \tag{14}$$

By guess, $x = 4$ is a solution of equation (14), we try to factorize $4x^3 - 20x^2 + x + 60$ by $(x - 4)$.

$$\begin{array}{r|rrr}
 & 4x^2 & -4x & -15 \\ \hline
x-4 & 4x^3 - 20x^2 & +x & +60 \\
 & -4x^3 + 16x^2 & & \\ \hline
 & -4x^2 & +x & \\
 & 4x^2 & -16x & \\ \hline
 & & -15x & +60 \\
 & & 15x & -60 \\ \hline
 & & & 0
\end{array}$$

Then,

$$\begin{aligned}
f'(x) = 4x^3 - 20x^2 + x + 60 &= 0 \\
\Rightarrow \quad (x-4)(4x^2 - 4x - 15) &= 0 \\
\Rightarrow \quad (x-4)(2x+3)(2x-5) &= 0 \\
\Rightarrow \quad x = 4 \quad \text{or} \quad x = -3/2 \quad \text{or} \quad x = 5/2 &
\end{aligned}$$

Check the second derivative, equation (12), when $x = 4$, $x = -3/2$, and $x = 5/2$ respectively.

$$f''(4) = 12 \times 4^2 - 40 \times 4 + 1 = 33 > 0$$

$$f''(-3/2) = 12 \times (-3/2)^2 - 40 \times (-3/2) + 1 = 88 > 0$$

$$f''(5/2) = 12 \times (5/2)^2 - 40 \times (5/2) + 1 = -24 < 0$$

In addition, $f(-3/2) < f(4)$. So $x = -3/2$ is a global minimum point. $x = 4$ is a local minimum point. $x = 5/2$ is local maximum point.

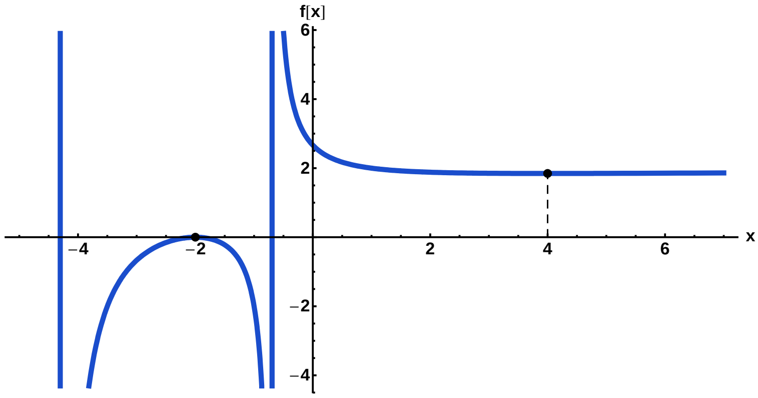
Set the second derivative, equation (12), to be zero.

$$\begin{aligned} f''(x) = 12x^2 - 40x + 1 &= 0 \\ \Rightarrow \qquad x &= \tfrac{40 \pm \sqrt{40^2 - 4 \times 12 \times 1}}{2 \times 12} \\ \Rightarrow \qquad x &= \tfrac{10 \pm \sqrt{97}}{6} \end{aligned}$$

Since the third derivative, equation (13), is zero only when $x = 40/24 = 5/3$. So $x = \frac{10 \pm \sqrt{97}}{6}$ are two inflection points.

b. $f(x) = \dfrac{2x^2+8x+8}{x^2+5x+3}$ You need not find the points of inflection for this problem. Hint: You should find the second derivative of f(x) but here is the answer: $-\frac{4(x^3-3x^2-24x-37)}{(x^2+5x+3)^3}$.

FIGURE 2. $f(x) = \dfrac{2x^2+8x+8}{x^2+5x+3}$



First, simply $f(x)$.

$$\begin{aligned} f(x) &= \frac{2x^2+8x+8}{x^2+5x+3} \\ &= \frac{2x^2+10x+6-2x+2}{x^2+5x+3} \\ &= 2-\frac{2(x-1)}{x^2+5x+3} \end{aligned}$$

Then the first and second derivative of $f(x)$ are given by

$$\begin{aligned} f'(x) &= -\frac{2(x^2+5x+3)-2(x-1)(2x+5)}{(x^2+5x+3)^2} \\ &= \frac{2x^2-4x-16}{(x^2+5x+3)^2} \\ &= \frac{2(x-4)(x+2)}{(x^2+5x+3)^2} \end{aligned}$$

Set the first derivative equal to zero.

$$f'(x) = \frac{2(x-4)(x+2)}{(x^2+5x+3)^2} = 0$$

$$\Rightarrow \qquad x = 4 \quad \text{or} \quad x = -2$$

We still need to check the second derivative when $x = 4$ and $x = -2$. The second derivative is given by

$$
\begin{aligned}
f''(x) &= \frac{d\left[\dfrac{2x^2-4x-16}{(x^2+5x+3)^2}\right]}{d\,x} \\
&= \frac{(4x-4)(x^2+5x+3)^2-(2x^2-4x-16)[2(x^2+5x+3)(2x+5)]}{(x^2+5x+3)^4} \\
&= \frac{(4x-4)(x^2+5x+3)-(2x^2-4x-16)(4x+10)}{(x^2+5x+3)^3} \\
&= \frac{4x^3-4x^2+20x^2-20x+12x-12-(8x^3-16x^2-64x+20x^2-40x-160)}{(x^2+5x+3)^3} \\
&= \frac{-4x^3+12x^2+96x+148}{(x^2+5x+3)^3} \\
&= -\frac{4(x^3-3x^2-24x-37)}{(x^2+5x+3)^3}
\end{aligned}
$$

Then,

$$
f''(4) = -\frac{4(4^3-3\times 4^2-24\times 4-37)}{(4^2+5\times 4+3)^3} = \frac{468}{39^3} = \frac{4}{507} > 0
$$

$$
f''(-2) = -\frac{4((-2)^3-3\times(-2)^2-24\times(-2)-37)}{((-2)^2+5\times(-2)+3)^3} = \frac{36}{(-3)^3} = -\frac{4}{3} < 0
$$

So $x = -2$ is a local maximum point. $x = 4$ is a local minimum point.

Problem 6. Solve the following system of equations.

$$360x_1^{-2/3}x_2^{2/5} - 160 = 0$$

$$432x_1^{1/3}x_2^{-3/5} - 162 = 0$$

$x_1 = 27.$

From the second equation,

$$\begin{aligned} 432x_1^{1/3}x_2^{-3/5} - 162 &= 0 \\ \Rightarrow \quad 432x_1^{1/3}x_2^{-3/5} &= 162 \\ \Rightarrow \quad \frac{432}{162}x_2^{-3/5} &= x_1^{-1/3} \\ \Rightarrow \quad x_1^{-1/3} &= \frac{8}{3}x_2^{-3/5} \\ \Rightarrow \quad x_1^{-2/3} &= \left(\frac{8}{3}\right)^2 x_2^{-6/5} \end{aligned}$$

Substitute $x_1^{-2/3} = \left(\frac{8}{3}\right)^2 x_2^{-6/5}$ into the first equation.

$$\begin{aligned} 360x_1^{-2/3}x_2^{2/5} - 160 &= 0 \\ \Rightarrow \quad 360\left(\left(\frac{8}{3}\right)^2 x_2^{-6/5}\right)x_2^{2/5} - 160 &= 0 \\ \Rightarrow \quad x_2^{-4/5} &= \frac{160}{360}\left(\frac{3}{8}\right)^2 = 2^{-4} \\ \Rightarrow \quad x_2^{1/5} &= 2 \\ \Rightarrow \quad x_2 &= 32 \end{aligned}$$

Substitute $x_2 = 32$ into $x_1^{-1/3} = \frac{8}{3}x_2^{-3/5}$.

$$\begin{aligned} x_1^{-1/3} &= \frac{8}{3}x_2^{-3/5} \\ \Rightarrow \quad x_1^{-1/3} &= \frac{8}{3}(32)^{-3/5} = 3^{-1} \\ \Rightarrow \quad x_1^{1/3} &= 3 \\ \Rightarrow \quad x_1 &= 27 \end{aligned}$$

So the solution is

$$x_1 = 27, \; x_2 = 32$$

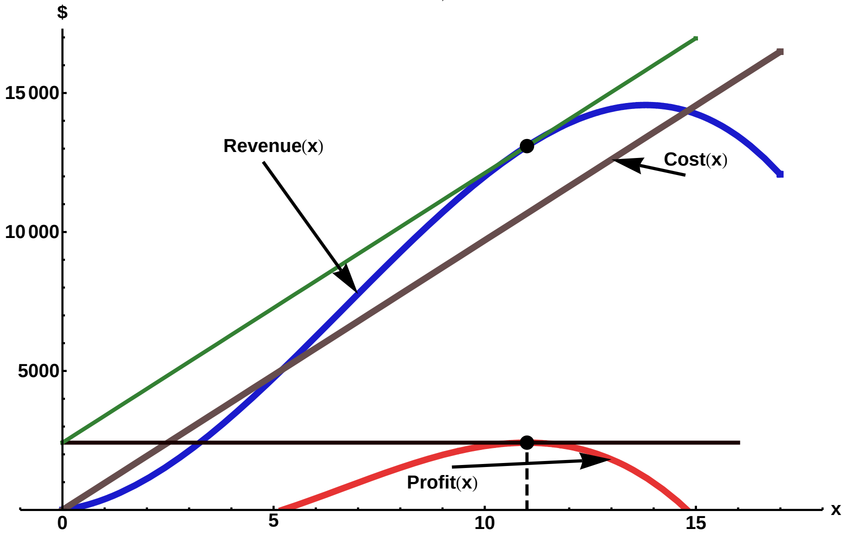
Problem 7. In the following problem you are given a production function for a firm where y is the level of output and x is the level of the variable input. You are given the price (p) of the output and the price (w) of the single variable input.

$$output\ price = p = 5$$

$$input\ price = w = 970$$

$$y \;=\; output \;=\; f(x) \;=\; 40x + 40x^2 - 2x^3$$

FIGURE 3. Revenue, Cost and Profit



a. Write down an equation that represents profit for the firm.

The profit is given by

$$\begin{aligned} Profit &= Revenue - Cost \\ &= py - wx \\ &= 5(40x + 40x^2 - 2x^3) - 970x \\ &= -10x^3 + 200x^2 - 770x \end{aligned}$$

b. Maximize this function by taking its derivative with respect to the variable input x and setting the resulting equation equal to zero.

$$
\begin{aligned}
& \frac{d\,Profit}{d\,x} = 0 \\
\Rightarrow\quad & \frac{d\left(-10x^3+200x^2-770x\right)}{d\,x} = 0 \\
\Rightarrow\quad & -30x^2+400x-770 = 0 \\
\Rightarrow\quad & 3x^2-40x+77 = 0 \\
\Rightarrow\quad & (x-11)(3x-7) = 0 \\
\Rightarrow\quad & x = 11 \quad \text{or} \quad x = 7/3
\end{aligned}
$$

c. If you identify more than one critical value from setting the first derivative of profit equal to zero, show which ones, if any, maximize profit.

Two points, $x = 11$ and $x = 7/3$, are identified from setting the first derivative of profit equal to zero. We need to check the second derivative. The second derivative is given by

$$
\frac{d^2\,Profit}{d\,x^2} = \frac{d\left(-30x^2+400x-770\right)}{d\,x}
$$

$$
= -60x+400
$$

When $x = 11$, $\frac{d^2\,Profit}{d\,x^2} = -60\times 11+400 = -260 < 0$. When $x = 7/3$, $\frac{d^2\,Profit}{d\,x^2} = -60\times 7/3+400 = 260 > 0$.

So $x = 11$ is the point where the profit attains its maximum.

d. Explain in words why the value of the marginal product for this firm is equal to the price of the single variable input at the profit maximizing level of input use. You can use the following information in explaining this phenomenon. Say something about the benefits of using an input not being less than the cost of the input.

$$\begin{aligned} Output =& y = f(x) \\ MP =& \text{Marginal Product} = \frac{d\,f(x)}{dx} = f'(x) = \frac{\Delta y}{\Delta x} \\ Revenue =& pf(x) \\ Cost =& wx \\ Profit = \pi =& Revenue - Cost = pf(x) - wx \\ \frac{d\pi}{dx} =& \end{aligned}$$

The first derivative of profit is given by

$$\frac{d\,Profit}{d\,x} = \frac{d\ (pf(x) - wx)}{d\,x} = pf'(x) - w$$

So setting the first derivative to zero in order to maximize the profit can be expressed by below equation.

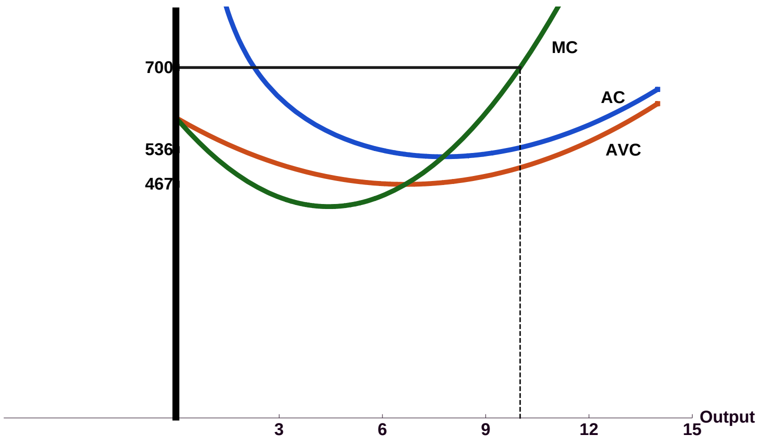
$$pf'(x) - w = 0 \tag{15}$$

In fact, the left side of equation (15) is the output price times the marginal product, i.e. the value of the marginal product. And the right side of equation (15) is the price of input variable. As a result, value of the marginal product for this firm is equal to the price of the single variable input at the profit maximizing level of input use.

In other words, when the value of the marginal product is higher than the price of input variable, the firm can profit from increasing the input. On the other hand, when the value of the marginal product is less than the price of input variable, the firm can profit from decreasing the input. Thus, when the value of the marginal product is equal to the price of input variable, the firm attains the optimal input level for maximizing profit.

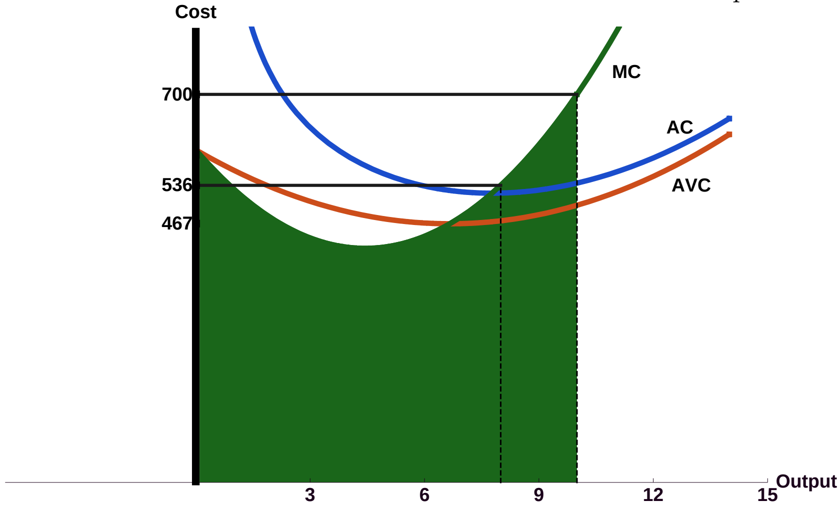
Problem 8. The cost function for a firm is a rule or mapping that tells the total cost of production of any output level produced by the firm. If the variable y represents the output of the firm, then the cost function is given by c(y). Marginal cost represents the change in the cost of production for the firm as output changes and is given by the derivative of the cost function with respect to output,i.e., Marginal Cost(MC) $= \frac{dc(y)}{dy}$. A competitive firm facing a fixed output price maximizes profit at the output level where marginal cost is equal to price as in the figure 4.

FIGURE 4. Profit Maximization



The area below the cost curve is a measure of variable cost and can be found by integrating the marginal cost curve from 0 to any given output level y. The shaded area in figure 5 represents the variable cost of production for the cost function $c(y) = 400 + 600y - 40\, y^2 + 3y^3$.

FIGURE 5. Variable Cost of Production and Producer Surplus



Producer surplus is the area below a given price and above the marginal cost curve. Producer surplus is the unshaded area below the horizontal line at 700 in figure 5. Producer surplus can be computed by subtracting the shaded area from total revenue.

a. Without writing down an equation for profit, find the profit maximizing level of output for the following firm.

$$price \;=\; p \;=\$700$$

$$cost \;=\; c(y) \;= 400 \;+\; 600y \;-\; 40y^2 \;+\; 3y^3$$

Given that you have no second order conditions from profit maximization per se, how do you know which level of output to choose? You might check the derivative of marginal cost and whether MC is upward or downward sloping at each point.

A competitive firm facing a fixed output price maximizes profit at the output level where marginal cost is equal to price. That is,

$$MC(y) = c'(y) = 600 - 80y + 9y^2 = 700 \tag{16}$$

And equation (16) has solutions given below.

$$\begin{aligned} 600 - 80y + 9y^2 &= 700 \\ \Rightarrow \quad 9y^2 - 80y - 100 &= 0 \\ \Rightarrow \quad (y-10)(9y+10) &= 0 \\ \Rightarrow \quad y = 10 \quad \text{or} \quad & y = -10/9 \end{aligned}$$

And the second derivative of marginal cost is given by

$$\frac{d\,MC(y)}{d\,y} = 18y - 80$$

When $y = 10$, the second derivative of marginal cost is $18 \times 10 - 80 = 100 > 0$. And when $y = -10/9$, the second derivative of marginal cost is $18 \times (-10/9) - 80 = -100 < 0$.

Thus the marginal cost is upward at $y = 10$ and downward at $y = -10/9$. So $y = 10$ is the point where the firm maximizes the profit.

b. Find the profit maximizing level of output for the same firm when the price is \$536.

When the price is \$536, the equation of maximizing profit is given by

$$\begin{aligned} 600 - 80y + 9y^2 &= 536 \\ \Rightarrow \quad 9y^2 - 80y + 64 &= 0 \\ \Rightarrow \quad (y-8)(9y-8) &= 0 \\ \Rightarrow \quad y = 8 \quad \text{or} \quad y &= 8/9 \end{aligned}$$

When $y = 8$, the second derivative of marginal cost is $18 \times 8 - 80 = 64 > 0$. And when $y = 8/9$, the second derivative of marginal cost is $18 \times 8/9 - 80 = -64 < 0$.

Thus the marginal cost is upward at $y = 8$ and downward at $y = 8/9$. So $y = 8$ is the point where the firm maximizes the profit.

c. Explain in words why setting price equal to marginal cost and solving for the optimal output y gives the same answers as taking the derivative of profit with respect to y, setting the result equal to zero and solving for the optimal y. Remember that

$$Profit = py - c(y)$$
$$Profit = 700y - \left[400 \; + \; 600y \; - \; 40\; y^2 \; + \; 3y^3\right]$$

The first derivative of profit is given by

$$\frac{d\,Profit}{d\,y} = p - c'(y) \tag{17}$$

So setting equation (17) to zero is to set $p = c'(y)$, i.e., set the price to marginal cost.

In other words, when the price is higher than the marginal cost, the firm can profit from increasing the output level. On the other hand, when the price is less than the marginal cost, the firm can profit from decreasing the output level. So the optimal level for maximizing the profit is the level where the marginal cost equals to the price.

d. Show that variable cost for this firm when it maximizes profit with a price of \$700 is \$5000.

The output level is $y = 10$ for this firm when it maximizes profit with a price of \$700. And the variable cost for this firm is given by

$$\int_0^{10} (600 - 80y + 9y^2)dy = \left(600y - 40y^2 + 3y^3\right)\Big|_0^{10}$$
$$= 5000$$

e. What is producer surplus for this profit maximizing firm when the price is \$700? The producer surplus is given by

$$Producer\ surplus = py - \int_0^{10} MC(y)dy$$
$$= 700 \times 10 - 5000$$
$$= 2000$$

f. Show that variable cost for this firm when it maximizes profit with a price of \$536 is \$3776.

The output level is $y = 8$ for this firm when it maximizes profit with a price of \$536. And the variable cost for this firm is given by

$$\int_0^{8} (600 - 80y + 9y^2)dy = \left(600y - 40y^2 + 3y^3\right)\Big|_0^{8}$$
$$= 3776$$

g. What is producer surplus for this profit maximizing firm when the price is \$536?

The producer surplus is given by

$$\begin{aligned} Producer\ surplus &= py - \int_0^8 MC(y)dy \\ &= 536 \times 8 - 3776 \\ &= 512 \end{aligned}$$

h. How much is the firm worse off when price falls from \$700 to \$536?

The decrease of the producer surplus when price falls from \$700 to \$536 is given by

$$2000 - 512 = 1488$$

So the firm is worse off \$1488 when price falls from \$700 to \$536.

i. Cross-hatch the change in producer surplus in Figure 5.

Figure 6. Change in producer surplus in Figure 5