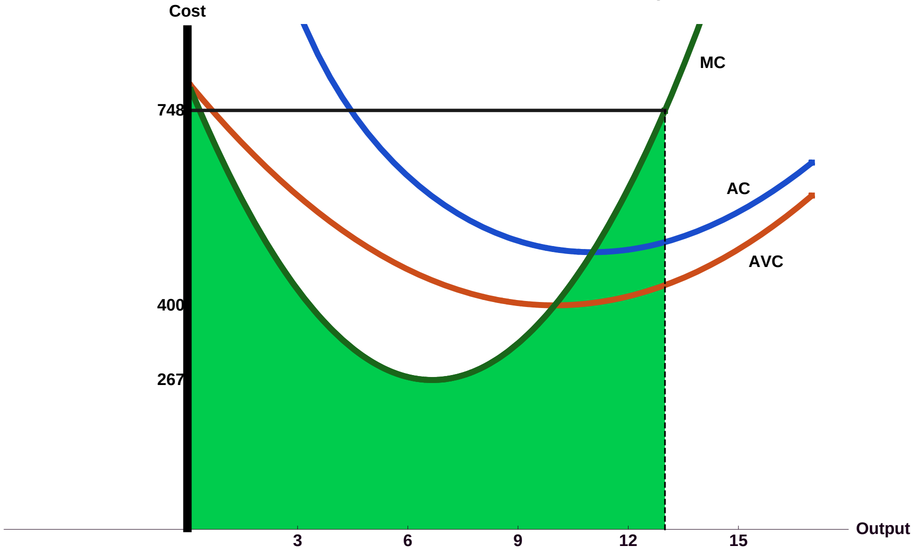
ECONOMICS 207
SPRING 2008
LABORATORY EXERCISE 13
KEY

Problem 1. The cost function for a firm is a rule or mapping that tells the minimum total cost of production of any output level produced by the firm for a fixed level of input prices. If the variable y represents the output of the firm, then the cost function is given by c(y,w) or c(y). Marginal cost represents the change in the cost of production for the firm as output changes and is given by the derivative of the cost function with respect to output, i.e., Marginal Cost(MC) = $\frac{dc(y)}{dy}$. A graph of price and marginal cost is given in figure 1.

FIGURE 1. Profit Maximization Given Marginal Cost



Consider a competitive firm with the following technology (as represented by its cost function) and output price.

$$
\begin{aligned}
price &= p = \$748 \\
cost &= c(y) = 1000 + 800y - 80y^2 + 4y^3
\end{aligned}
$$

Date: August 6, 2008.

a. Write down an equation for profit for this firm and find the potential levels of output that maximize profit.

The profit is given by

$$\begin{aligned} Profit &= Revenue - Cost \\ &= py - c(y) \\ &= 748y - \left(1000 + 800y - 80y^2 + 4y^3\right) \\ &= -4y^3 + 80y^2 - 52y - 1000 \end{aligned}$$

The the first and second derivative of profit with respect to y are given by

$$\begin{aligned} \frac{d\,Profit}{d\,y} &= \frac{d\,\left(-4y^3 + 80y^2 - 52y - 1000\right)}{d\,y} \\ &= -12y^2 + 160y - 52 \end{aligned}$$

$$\frac{d^2\,Profit}{d\,y^2} = -24y + 160$$

To maximize profit, set the first derivative of profit with respect to y to be zero.

$$\begin{aligned} & \frac{d\,Profit}{d\,y} &= 0 \\ \Rightarrow & -12y^2 + 160y - 52 &= 0 \\ \Rightarrow & -(y-13)(12y-4) &= 0 \\ \Rightarrow & y = 13 \quad \text{or} \quad y = 1/3 \end{aligned}$$

b. Using the second order conditions from profit maximization, determine which of the levels of output from part a that maximize profit?

Check the second derivative of profit with respect to y.
When $y = 13$,

$$-24y + 160 = -24 \times 13 + 160 = -152 < 0$$

When $y = 1/3$,

$$-24y + 160 = -24 \times (1/3) + 160 = 152 >$$

So output level $y = 13$ is the level of maximizing profit.

c. Explain in words why setting price equal to marginal cost and solving for the optimal output y gives the same answers as taking the derivative of profit with respect to y, setting the result equal to zero and solving for the optimal y. Remember that

$$Profit = py - c(y)$$

$$Profit = 748y - \left[1000 + 800y - 80\,y^2 + 4y^3\right]$$

Setting the first derivative of profit with respect to y to zero is expressed by below equation.

$$\frac{d\,Profit}{d\,y} = p - c'(y) = p - MC(y) = 0,$$

which is equivalent to

$$p = MC(y)$$

So setting price equal to marginal cost and solving for the optimal output y gives the same answers as setting the first derivative of profit with respect to y to zero.

In other words, when the price of output, p, is higher than the marginal cost, the firm could profit by increasing the output level. On the other hand, when the price of output, p, is less than the marginal cost, the firm could profit by decreasing the output level. As a result, the firm will maximize the profit when the price of output, p, is equal to the marginal cost.

d. Consider a competitive firm with the following technology (as represented by its cost function) and output price.

$$price = p = \$748$$

$$cost = c(y) = 1000 + 800y - 80y^2 + 4y^3$$

Without writing down an equation for profit, find the levels of output which potentially maximize profit using what you have learned in general about profit maximization for a competitive firm.

Setting the marginal cost equal to the price of output.

$$\begin{aligned} MC(y) = c'(y) = 800 - 160y + 12y^2 &= 748 \\ \Rightarrow \quad 52 - 160y + 12y^2 &= 0 \\ \Rightarrow \quad (y-13)(12y-4) &= 0 \\ \Rightarrow \quad y = 13 \quad &\text{or} \quad x = \tfrac{1}{3} \end{aligned}$$

e. Given that you have no second order conditions from profit maximization per se, make a coherent argument about which of the two potential output levels you found in part d maximizes profit.

In order to maximize the profit, the marginal cost, $MC(y)$, should be upward at the optimal point. That is, when the marginal cost is less than the price p for output level that is less than the optimal level; and the marginal cost is higher than the price p for output level that is higher than the optimal level. Thus check the first derivative of $MC(y)$.

$$\frac{d\,MC(y)}{d\,y} = -160 + 24y$$

When $y = 13$, $-160 + 24y = 152 > 0$; when $y = 1/3$, $-160 + 24y = -152 < 0$. So $y = 13$ is the optimal output level for profit maximization.

f. Integrate the marginal cost function you found in part d to obtain the variable cost function and then use it to show that the level of variable cost for this firm when it maximizes profit with a price of \$748 is \$5668.

By the method given in the problem, the variable cost function is given by

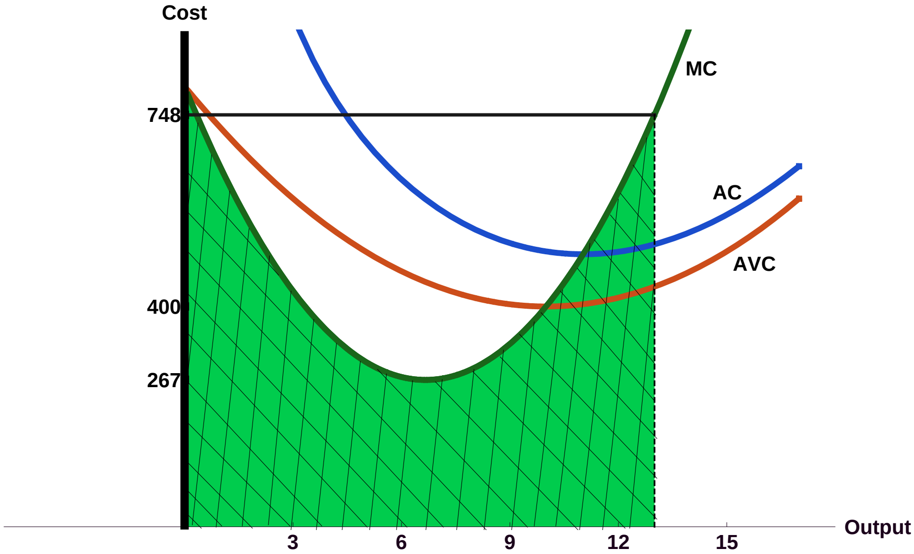
$$\begin{aligned} VC(y) &= \int_0^y MC(y)dy \\ &= \int_0^y \left(800 - 160y + 12y^2\right) dy \\ &= 800y - 80y^2 + 4y^3 \end{aligned}$$

The output level is that $y = 13$ for this firm when it maximizes profit with a price of \$748. So the variable cost of this firm for $y = 13$ is given by

$$\begin{aligned} VC(13) &= 800 \times 13 - 80 \times 13^2 + 4 \times 13^3 \\ &= 5668 \end{aligned}$$

g. Cross-hatch the area represented by variable cost in part f and explain why the integral of the marginal cost function or the area under the marginal cost curve represents variable cost.

FIGURE 2. Variable cost



h. Given that revenue for this profit maximizing firm is $9724, what is producer surplus?

The producer surplus is given by

$$
\begin{aligned}
Producer\ surplus &= Revenue - Variable\ cost \\
&= 9724 - 5668 = 4056
\end{aligned}
$$

i. Shade this level of producer surplus in Figure 1.

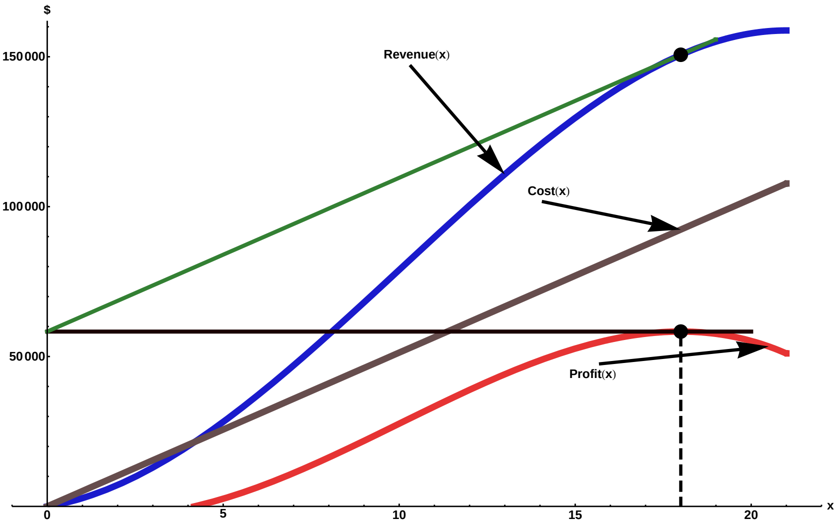
Problem 2. In the following problem you are given a production function for a firm where y is the level of output and x is the level of the variable input. You are given the price (p) of the output and the price (w) of the single variable input.

$$output\ price = p = 10$$

$$input\ price = w = 5130$$

$$y = output = f(x) = 189x + 90x^2 - 3x^3$$

Figure 3. Revenue, Cost and Profit



a. Find values of x that potentially maximize **output** for this firm.

Set the first derivative of output with respect to x to be zero.

$$\begin{aligned} \frac{dy}{dx} = 189 + 180x - 9x^2 &= 0 \\ \Rightarrow \quad 21 + 20x - x^2 &= 0 \\ \Rightarrow \quad (x-21)(x+1) &= 0 \\ \Rightarrow \quad x = 21 & \quad \text{or} \quad x = -1 \end{aligned}$$

b. Show which values of x in part a actually maximize output.

Check the second derivative of output with respect to x.

$$\frac{d^2 y}{d x^2} = 180 - 18x$$

When $x = 21$, $180 - 18x = 180 - 18 \times 21 = -198 < 0$;
when $x = -1$, $180 - 18x = 180 - 18 \times (-1) = 198 > 0$.
So $x = 11$ actually maximize the output level.

c. Write down an equation that represents profit for the firm.

The profit is given by

$$\begin{aligned} Profit &= Revenue - Cost = py - wx \\ &= 10(189x + 90x^2 - 3x^3) - 5130x \\ &= -30x^3 + 900x^2 - 3240x \end{aligned}$$

d. Maximize this function by taking its derivative with respect to the variable input x and setting the resulting equation equal to zero.

The first derivative of the profit function is given by

$$\frac{d\,Profit}{d\,x} = -90x^2 + 1800x - 3240 \tag{1}$$

$$\frac{d^2\,Profit}{d\,x^2} = -180x + 1800 = -180(x - 10) \tag{2}$$

Set (1) to be zero.

$$\begin{aligned} -90x^2 + 1800x - 3240 &= 0 \\ \Rightarrow \quad x^2 - 20x + 36 &= 0 \\ \Rightarrow \quad (x - 18)(x - 2) &= 0 \\ \Rightarrow \quad x = 18 \quad \text{or} \quad x = 2 \end{aligned}$$

e. If you identify more than one critical value from setting the first derivative of profit equal to zero, show which ones, if any, maximize profit.

Check the second derivative, equation (2), when $x = 18$ and $x = 2$ respectively.
When $x = 18$,

$$-180(x - 10) = -180 \times 8 < 0$$

When $x = 2$,

$$-180(x - 10) = -180 \times (-8) > 0$$

So when the input level $x = 18$, the profit attains its maximum.

f. Explain in words why the value of the marginal product for this firm is equal to the price of the single variable input at the profit maximizing level of input use. You can use the following information in explaining this phenomenon. Say something about the benefits of using an input not being less than the cost of the input.

$$\begin{aligned}
Output =& y = f(x) \\
MP =& \text{Marginal Product} = \frac{d\,f(x)}{dx} = f'(x) = \frac{\Delta y}{\Delta x} \\
Revenue =& pf(x) \\
Cost =& wx \\
Profit = \pi =& Revenue - Cost = pf(x) - wx \\
\frac{d\pi}{dx} =&
\end{aligned}$$

Setting the first derivative of profit with respect to input level x to be zero can be express by below equation.

$$\frac{d\,Profit}{d\,x} = pf'(x) - w = 0$$

And above equation is equivalent to $pf'(x) = w$, i.e., the value of the marginal product is equal to the price of the single variable input at the profit maximization.

In other words, if the value of the marginal product is higher than the cost of the input, the firm can profit more by increasing the input level. On the other hand, if the value of the marginal product is less than the cost of the input, the firm can profit more by decreasing the input level. As a result, the input level where the value of the marginal product is equal to the cost of the input is the level of maximizing profit.

g. Using the information from part 2f, explain why the slope of total revenue in figure 3 is equal to the slope of total cost at the profit maximizing level of input use.

The slope of total cost, wx, is given by

$$\frac{d\,wx}{d\,x} = w$$

And the slope of the total revenue is given by

$$\frac{d\,pf(x)}{d\,x} = pf'(x)$$

So the slope of total cost is the price of input; the slope of the total revenue is the value of marginal product. From part 2f, we know that the value of the marginal product is equal to the input price at the input level of maximizing profit. Then the slope of total revenue in figure 3 is equal to the slope of total cost at the profit maximizing level of input use.

Problem 3. For each of the following problems, write an equation that represents profit as a function of the two inputs x_1 and x_2.

Write it in the form π = pf(x_1,x_2)-w_1x_1-w_2x_2 and then simplify the expression. Then find all first and second partial derivatives of the function. Then set the partial derivatives with respect to x_1 and x_2 equal to zero and solve the equations for the levels of x_1 and x_2 that maximize profit. Then show that the level you found actually maximizes profit.

a.

$$f(x_1, x_2) = 10x_1 + 40x_2 - x_1^2 + x_1x_2 - x_2^2$$

$$p = 4$$

$$w_1 = 60, \qquad w_2 = 24$$

$$\begin{aligned} \pi &= 4(10x_1 + 40x_2 - x_1^2 + x_1x_2 - x_2^2) - 60x_1 - 24x_2 \\ &= 40x_1 + 160x_2 - 4x_1^2 + 4x_1x_2 - 4x_2^2 - 60x_1 - 24x_2 \\ &= -20x_1 + 136x_2 - 4x_1^2 + 4x_1x_2 - 4x_2^2 \end{aligned} \tag{3}$$

$\frac{\partial \pi}{\partial x_1} = -20 - 8x_1 + 4x_2$	$\frac{\partial \pi}{\partial x_2} = 136 + 4x_1 - 8x_2$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -8$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 4$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 4$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -8$

Find potential profit maximizing levels of x_1 and x_2.
We maximize profit by taking the derivatives of (3) setting them equal to zero and solving for x_1 and x_2.

$$\frac{\partial \pi}{\partial x_1} = -20 - 8x_1 + 4x_2 = 0 \tag{4a}$$

$$\frac{\partial \pi}{\partial x_2} = 136 + 4x_1 - 8x_2 = 0 \tag{4b}$$

Multiply equation 4a by 2 and add to 4b to obtain

$$\begin{aligned} -40 - 16x_1 \quad +8x_2 &= 0 \\ 136 + 4x_1 \quad -8x_2 &= 0 \\ \Rightarrow 96 - 12x_1 \quad +0x_2 &= 0 \\ \Rightarrow \qquad x_1 &= 8 \end{aligned} \tag{5}$$

Now substitute x_1 in equation 4a and solve for x_2.

$$\begin{aligned} -20 - 8x_1 + 4x_2 &= 0 \\ \Rightarrow -20 - 8(8) + 4x_2 &= 0 \\ \Rightarrow 4x_2 &= 84 \\ \Rightarrow x_2 &= 21 \end{aligned} \tag{6}$$

By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

$$\begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -8 & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 4 \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 4 & \frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -8 \end{vmatrix} = 64 - 16 = 48 > 0.$$

Both diagonal elements are negative and the determinant of the Hessian is positive, so the input levels $x_1 = 8,\ x_2 = 21$ represent a point of profit maximization.

b.

$$
\begin{aligned}
f(x_1, x_2) &= 30x_1 \; + \; 15x_2 \; - \; 2x_1^2 \; + \; x_1x_2 \; - \; x_2^2 \\
p &= 4 \\
w_1 &= 60, \qquad w_2 \; = \; 12 \\
x_1 &= 6, \qquad x_2 \; = \; 9
\end{aligned}
$$

$$
\begin{aligned}
\pi &= 4(30x_1 + 15x_2 - 2x_1^2 + x_1x_2 - x_2^2) - 60x_1 - 12x_2 \\
&= 60x_1 + 48x_2 - 8x_1^2 + 4x_1x_2 - 4x_2^2
\end{aligned} \tag{7}
$$

$\frac{\partial \pi}{\partial x_1} = 60 - 16x_1 + 4x_2$	$\frac{\partial \pi}{\partial x_2} = 48 + 4x_1 - 8x_2$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -16$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 4$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 4$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -8$

Find potential profit maximizing levels of x_1 and x_2.
We maximize profit by taking the derivatives of (7) and setting them equal to zero for x_1 and x_2.

$$\frac{\partial \pi}{\partial x_1} = 60 - 16x_1 + 4x_2 = 0 \tag{8a}$$

$$\frac{\partial \pi}{\partial x_2} = 48 + 4x_1 - 8x_2 = 0 \tag{8b}$$

Multiply equation 8a by 2 and add it to 8b to obtain

$$\begin{aligned} 120 - 32x_1 + 8x_2 &= 0 \\ 48 + 4x_1 - 8x_2 &= 0 \\ \Rightarrow \quad 168 - 28x_1 &= 0 \\ \Rightarrow \quad x_1 &= 168/28 = 6 \end{aligned}$$

Multiply equation 8b by 4 and add it to 8a to obtain

$$\begin{aligned} 60 - 16x_1 + 4x_2 &= 0 \\ 192 + 16x_1 - 32x_2 &= 0 \\ \Rightarrow \quad 252 - 28x_2 &= 0 \\ \Rightarrow \quad x_2 &= 252/28 = 9 \end{aligned}$$

By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

$$\begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -16 & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 4 \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 4 & \frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -8 \end{vmatrix} = 128 - 16 = 112 > 0.$$

Both diagonal elements are negative and the determinant of the Hessian is positive, so the input levels $x_1 = 6$, $x_2 = 9$ represent a point of profit maximization.

c.

$$
\begin{aligned}
f(x_1, x_2) &= 40x_1 \; + \; 10x_2 \; - \; 2x_1^2 \; + \; 2x_1x_2 \; - \; x_2^2 \\
p &= 1 \\
w_1 &= 34, \qquad w_2 \; = \; 2 \\
x_1 &= 7, \qquad x_2 \; = \; 11
\end{aligned}
$$

$$
\begin{aligned}
\pi &= 1(40x_1 + 10x_2 - 2x_1^2 + 2x_1x_2 - x_2^2) - 34x_1 - 2x_2 \\
&= 6x_1 + 8x_2 - 2x_1^2 + 2x_1x_2 - x_2^2
\end{aligned} \tag{9}
$$

$\frac{\partial \pi}{\partial x_1} = 6 - 4x_1 + 2x_2$	$\frac{\partial \pi}{\partial x_2} = 8 + 2x_1 - 2x_2$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -4$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 2$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 2$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} - 2$

Find potential profit maximizing levels of x_1 and x_2.
We maximize profit by taking the derivatives of (9) and setting them equal to zero for x_1 and x_2.

$$\frac{\partial \pi}{\partial x_1} = 6 - 4x_1 + 2x_2 = 0 \tag{10a}$$

$$\frac{\partial \pi}{\partial x_2} = 8 + 2x_1 - 2x_2 = 0 \tag{10b}$$

Add equation 10a to 10b.

$$\begin{aligned} 6 - 4x_1 + 2x_2 &= 0 \\ 8 + 2x_1 - 2x_2 &= 0 \\ \Rightarrow \quad 14 - 2x_1 &= 0 \\ \Rightarrow \quad x_1 &= 7 \end{aligned}$$

Add equation 10b multiplied by 2 to 10a.

$$\begin{aligned} 6 - 4x_1 + 2x_2 &= 0 \\ 16 + 4x_1 - 4x_2 &= 0 \\ \Rightarrow \quad 22 - 2x_2 &= 0 \\ \Rightarrow \quad x_2 &= 11 \end{aligned}$$

By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

$$\begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -4 & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 2 \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 2 & \frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -2 \end{vmatrix} = 8 - 4 = 4 > 0.$$

Both diagonal elements are negative and the determinant of the Hessian is positive, so the input levels $x_1 = 7,\ x_2 = 11$ represent a point of profit maximization.

d.

$$\begin{aligned} f(x_1, x_2) &= x_1^{1/2} x_2^{1/3} \\ p &= 594 \\ w_1 &= 81, \qquad w_2 \;=\; 242 \\ x_1 &= 121, \qquad x_2 \;=\; 27 \end{aligned}$$

$$\pi \;=\; 594 x_1^{1/2} x_2^{1/3} - 81 x_1 - 242 x_2 \tag{11}$$

$\frac{\partial \pi}{\partial x_1} = 297 x_1^{-1/2} x_2^{1/3} - 81$	$\frac{\partial \pi}{\partial x_2} = 198 x_1^{1/2} x_2^{-2/3} - 242$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -\frac{297}{2} x_1^{-3/2} x_2^{1/3}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 99 x_1^{-1/2} x_2^{-2/3}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 99 x_1^{-1/2} x_2^{-2/3}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} - 132 x_1^{-1/2} x_2^{-5/3}$

Find potential profit maximizing levels of x_1 and x_2.
From equation (11) we have

$$297x_1^{-1/2}x_2^{1/3} - 81 = 0 \tag{11.1}$$

$$198x_1^{1/2}x_2^{-2/3} - 242 = 0 \tag{11.2}$$

Rearrange the first equation 11.1 to obtain

$$\begin{aligned}
x_1^{-1/2}x_2^{1/3} &= \frac{81}{297} = \frac{3}{11} \\
\Rightarrow x_1^{1/2}x_1^{-1/2}x_2^{1/3} &= \frac{3}{11}x_1^{1/2} \\
\Rightarrow x_2^{1/3} &= \frac{3}{11}x_1^{1/2} \\
\Rightarrow x_2 &= \left(\frac{3}{11}\right)^3 \left(x_1^{1/2}\right)^3 \\
&= \left(\frac{3}{11}\right)^3 x_1^{3/2}
\end{aligned} \tag{11.1.a}$$

Rearrange the second equation 11.2 slightly to obtain

$$x_1^{1/2}x_2^{-2/3} = \frac{242}{198} = \frac{11}{9} \tag{11.2$'$}$$

Now substitute x_2 from equation 11.1.a into equation 11.2′ to obtain

$$\begin{aligned}
x_1^{1/2}\left(\left(\frac{3}{11}\right)^3 x_1^{3/2}\right)^{-2/3} &= \frac{11}{9} \\
\Rightarrow x_1^{1/2}\left(\frac{3}{11}\right)^{-2} x_1^{-1} &= \frac{11}{9} \\
\Rightarrow x_1^{-1/2}\left(\frac{3}{11}\right)^{-2} &= \frac{11}{9} \\
\Rightarrow x_1^{-1/2} &= \frac{11}{9}\left(\frac{3}{11}\right)^2 \\
\Rightarrow x_1 &= \left(\frac{11}{9}\left(\frac{3}{11}\right)^2\right)^{-2} = \left(\frac{11}{9}\right)^{-2}\left(\frac{3}{11}\right)^{-4} \\
&= 11^{-2}3^4 3^{-4}11^4 = 11^2 = 121
\end{aligned} \tag{11.2.a}$$

Now substitute x_1 from equation 11.2.a into equation 11.1.a to obtain

$$
\begin{aligned}
x_2 &= \left(\frac{3}{11}\right)^3 x_1^{3/2} \\
&= \left(\frac{3}{11}\right)^3 (121)^{3/2} \\
&= 3^3\ 11^{-3}\ (1)^3 \ = \ 3^3 \ = \ 27
\end{aligned}
$$

By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.
Remember that $x_1 = 121, \quad x_2 = 27$. Substituting in the Hessian matrix we obtain the following.

$$\begin{vmatrix}
\begin{aligned}\frac{\partial^2 \pi}{\partial x_1 \partial x_1} &= -\frac{297}{2} x_1^{-3/2} x_2^{1/3} \\ &= -\frac{297}{2}(121)^{-3/2}(27)^{1/3} \\ &= -\frac{3^3 \times 11}{2} 11^{-3} \times 3 \\ &= -\frac{3^4}{2 \times 11^2} \\ &= -\frac{81}{242}\end{aligned} & \begin{aligned}\frac{\partial^2 \pi}{\partial x_1 \partial x_2} &= 99 x_1^{-1/2} x_2^{-2/3} \\ &= 99(121)^{-1/2}(27)^{-2/3} \\ &= 3^2 \times 11 \times 11^{-1} \times 3^{-2} \\ &= 1\end{aligned} \\
\begin{aligned}\frac{\partial^2 \pi}{\partial x_2 \partial x_1} &= 99 x_1^{-1/2} x_2^{-2/3} \\ &= 99(121)^{-1/2}(27)^{-2/3} \\ &= 3^2 \times 11 \times 11^{-1} \times 3^{-2} \\ &= 1\end{aligned} & \begin{aligned}\frac{\partial^2 \pi}{\partial x_2 \partial x_2} &= -132 x_1^{1/2} x_2^{-(5/3} = \frac{-484}{81} \\ &= -132(121)^{1/2}(27)^{-5/3} \\ &= -2^2 \times 3 \times 11 \times 11 \times 3^{-5} \\ &= -\frac{4 \times 121}{3^4} \\ &= -\frac{484}{81}\end{aligned}
\end{vmatrix} = \left(\frac{-81}{242}\right)\left(\frac{-484}{81}\right) - (1)(1) = 2 - 1 = 1 > 0.$$

Both diagonal elements are negative and the determinant of the Hessian is positive, so the input levels $x_1 = 121, \; x_2 = 27$ represent a point of profit maximization.

e.

$$\begin{aligned} f(x_1, x_2) &= x_1^{3/5} x_2^{1/3} \\ p &= 240 \\ w_1 &= 64, \qquad w_2 \;=\; 135 \\ x_1 &= 243, \qquad x_2 \;=\; 64 \end{aligned}$$

$$\pi = 240x_1^{3/5}x_2^{1/3} - 64x_1 - 135x_2 \tag{12}$$

$\frac{\partial \pi}{\partial x_1} = 144x_1^{-2/5}x_2^{1/3} - 64$	$\frac{\partial \pi}{\partial x_2} = 80x_1^{3/5}x_2^{-2/3} - 135$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -\frac{288}{5}x_2^{1/3}x_1^{-7/5}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 48x_1^{-2/5}x_2^{-2/3}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 48x_1^{-2/5}x_2^{-2/3}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -\frac{160}{3}x_1^{3/5}x_2^{-5/3}$

Find potential profit maximizing levels of x_1 and x_2.
By setting the first derivative of equation (12) to zero, we have

$$144x_1^{-2/5}x_2^{1/3} - 64 = 0 \tag{12.1}$$

$$80x_1^{3/5}x_2^{-2/3} - 135 = 0 \tag{12.2}$$

Rearrange equation (12.1), we have

$$\begin{aligned} 144x_1^{-2/5}x_2^{1/3} - 64 &= 0 \\ \Rightarrow \quad 144x_1^{-2/5} &= 64x_2^{-1/3} \\ \Rightarrow \quad x_2^{-1/3} &= \tfrac{144}{64}x_1^{-2/5} = \left(\tfrac{3}{2}\right)^2 x_1^{-2/5} \\ \Rightarrow \quad x_2^{-2/3} &= \left(\tfrac{3}{2}\right)^4 x_1^{-4/5} \end{aligned}$$

Substitute $x_2^{-2/3} = \left(\frac{3}{2}\right)^4 x_1^{-4/5}$ into equation (12.2).

$$\begin{aligned} 80x_1^{3/5}x_2^{-2/3} - 135 &= 0 \\ \Rightarrow \quad 80x_1^{3/5}\left(\tfrac{3}{2}\right)^4 x_1^{-4/5} - 135 &= 0 \\ \Rightarrow \quad x_1^{-1/5} &= \tfrac{135}{80}\left(\tfrac{2}{3}\right)^4 = 3^{-1} \\ \Rightarrow \quad x_1 &= 3^5 = 243 \end{aligned}$$

Substitute $x_1 = 243$ into $x_2^{-1/3} = \left(\frac{3}{2}\right)^2 x_1^{-2/5}$.

$$\begin{aligned} x_2^{-1/3} &= \left(\tfrac{3}{2}\right)^2 x_1^{-2/5} \\ \Rightarrow \quad x_2^{-1/3} &= \left(\tfrac{3}{2}\right)^2 243^{-2/5} = \tfrac{1}{4} \\ \Rightarrow \quad x_2 &= 64 \end{aligned}$$

In this table fill in values of x_1 and x_2 given to obtain numerical answers for the Hessian matrix.

$\frac{\partial \pi}{\partial x_1} = 0$	$\frac{\partial \pi}{\partial x_2} = 0$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -\frac{288}{5} x_2^{1/3} x_1^{-7/5}$ $= -\frac{288}{5} 64^{1/3} 243^{-7/5} = \frac{288}{5} \times (4) \times (3^{-7})$ $= -\frac{128}{1215}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 48 x_1^{-2/5} x_2^{-2/3}$ $= 48 \times 243^{-2/5} 64^{-2/3}$ $= \frac{1}{3}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = \frac{1}{3}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -\frac{45}{32}$

By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.
Remember that $x_1 = 243, \quad x_2 = 64$. Substituting in the Hessian matrix using preceding results, we obtain the following.

$$\begin{vmatrix} \frac{\partial^2\pi}{\partial x_1 \partial x_1} = -\frac{288}{5} x_2^{1/3} x_1^{-7/5} = -\frac{128}{1215} & \frac{\partial^2\pi}{\partial x_1 \partial x_2} = 48 x_1^{-2/5} x_2^{-2/3} = \frac{1}{3} \\ \frac{\partial^2\pi}{\partial x_2 \partial x_1} = 48 x_1^{-2/5} x_2^{-2/3} = \frac{1}{3} & \frac{\partial^2\pi}{\partial x_2 \partial x_2} = -\frac{45}{32} \end{vmatrix} = -\frac{128}{1215} \times \left(-\frac{45}{32}\right) - \frac{1}{3} \times \frac{1}{3} = \frac{4}{27} - \frac{1}{9}$$

$$= \frac{1}{27} > 0$$

Both diagonal elements are negative and the determinant of the Hessian is positive, so the input levels $x_1 = 243$, $x_2 = 64$ represent a point of profit maximization.

f.

$$\begin{aligned} f(x_1, x_2) &= x_1^{1/4} x_2^{3/7} \\ p &= 224 \\ w_1 &= 7, \qquad w_2 \; = \; 24 \\ x_1 &= 256, \qquad x_2 \; = \; 128 \end{aligned}$$

$$\pi = 224 x_1^{1/4} x_2^{3/7} - 7x_1 - 24x_2 \tag{13}$$

$\frac{\partial \pi}{\partial x_1} = 56 x_1^{-3/4} x_2^{3/7} - 7$	$\frac{\partial \pi}{\partial x_2} = 96 x_1^{1/4} x_2^{-4/7} - 24$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -42 x_1^{-7/4} x_2^{3/7}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 24 x_1^{-3/4} x_2^{-4/7}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 24 x_1^{-3/4} x_2^{-4/7}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -\frac{384}{7} x_1^{1/4} x_2^{-11/7}$

Find potential profit maximizing levels of x_1 and x_2.
By setting the first derivative of equation (13) to zero, we have

$$56x_1^{-3/4}x_2^{3/7} - 7 = 0 \tag{13.1}$$

$$96x_1^{1/4}x_2^{-4/7} - 24 = 0 \tag{13.2}$$

Rearrange equation (13.2), we have

$$\begin{aligned} 96x_1^{1/4}x_2^{-4/7} - 24 &= 0 \\ \Rightarrow \quad (96/24)x_1^{1/4}x_2^{-4/7} &= 1 \\ \Rightarrow \quad 4x_2^{-4/7} &= x_1^{-1/4} \\ \Rightarrow \quad x_1^{-3/4} &= 4^3x_2^{-12/7} \end{aligned}$$

Substitute $x_1^{-3/4} = 4^3x_2^{-12/7}$ into equation (13.1).

$$\begin{aligned} 56x_1^{-3/4}x_2^{3/7} - 7 &= 0 \\ \Rightarrow \quad 56\left(4^3x_2^{-12/7}\right)x_2^{3/7} &= 7 \\ \Rightarrow \quad x_2^{-9/7} &= 7/(56 \times 4^3) = 2^{-9} \\ \Rightarrow \quad x_2^{1/7} &= 7/(56 \times 4^3) = 2 \\ \Rightarrow \quad x_2 &= 7/(56 \times 4^3) = 128 \end{aligned}$$

Substitute $x_2 = 128$ into $x_1^{-1/4} = 4x_2^{-4/7}$.

$$\begin{aligned} x_1^{-1/4} &= 4x_2^{-4/7} \\ \Rightarrow \quad x_1^{1/4} &= x_2^{4/7}/4 \\ \Rightarrow \quad x_1^{1/4} &= 128^{4/7}/4 = 4 \\ \Rightarrow \quad x_1 &= 256 \end{aligned}$$

In this table fill in values of x_1 and x_2 given to obtain numerical answers for the Hessian matrix.

$\frac{\partial \pi}{\partial x_1} = 0$	$\frac{\partial \pi}{\partial x_2} = 0$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -\frac{21}{1024}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 24 x_1^{-3/4} x_2^{-4/7}$ $= 24 \times \left(256^{-3/4}\right)\left(128^{-4/7}\right)$ $= 24 \times 64^{-1} \times 16^{-1} = \frac{3}{128}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = \frac{3}{128}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -\frac{3}{28}$

By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.
Remember that $x_1 = 256, \quad x_2 = 128$. Substituting in the Hessian matrix using preceding results, we obtain the following.

$$\begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -\frac{21}{1024} & \begin{aligned} \frac{\partial^2 \pi}{\partial x_1 \partial x_2} &= 24 x_1^{-3/4} x_2^{-4/7} \\ &= \frac{3}{128} \end{aligned} \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} = \frac{3}{128} & \frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -\frac{3}{28} \end{vmatrix} = -\frac{21}{1024} \times \left(-\frac{3}{28}\right) - \frac{3}{128} \times \frac{3}{128}$$

$$= \frac{9}{4096} - \frac{9}{16384}$$

$$= \frac{27}{16384} > 0$$

Both diagonal elements are negative and the determinant of the Hessian is positive, so the input levels $x_1 = 256, \ x_2 = 128$ represent a point of profit maximization.

Problem 4. Find the listed partial derivatives of each of the following functions.

a. $\mathcal{L}(x_1, x_2, \lambda) \;=\; 60x_1 + 24x_2 - \lambda\left(10x_1 + 40x_2 - x_1^2 + x_1x_2 - x_2^2 - 583\right)$

$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 60 - 10\lambda + 2\lambda x_1 - \lambda x_2$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = 24 - 40\lambda - \lambda x_1 + 2\lambda x_2$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = -10x_1 - 40x_2 + x_1^2 - x_1x_2 + x_2^2 + 583$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = 2\lambda$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = -\lambda$	-$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = 10 - 2x_1 + x_2$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = -\lambda$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = 2\lambda$	- $\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = 40 + x_1 - 2x_2$
- $\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = 10 - 2x_1 + x_2$	-$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = 40 + x_1 - 2x_2$	-$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0$

b. $\mathcal{L}(x_1, x_2, \lambda) = 45x_1 + 15x_2 - 2x_1^2 + 2x_1x_2 - x_2^2 - \lambda\,(150x_1 + 10x_2 - 3590)$

$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 45 - 4x_1 + 2x_2 - 150\lambda$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = 15 + 2x_1 - 2x_2 - 10\lambda$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = -150x_1 - 10x_2 + 3590$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -4$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = 2$	$-\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = 150$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = 2$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -2$	$-\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = 10$
$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = 150$	$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = 10$	$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0$

c. $\mathcal{L}(x_1, x_2, \lambda) \;=\; 81x_1 + 242x_2 - \lambda\left(x_1^{1/2}x_2^{1/3} - 33\right)$

$\dfrac{\partial\mathcal{L}(x_1,x_2,\lambda)}{\partial x_1} = 81 - \frac{1}{2}\lambda x_1^{-1/2}x_2^{1/3}$	$\dfrac{\partial\mathcal{L}(x_1,x_2,\lambda)}{\partial x_2} = 242 - \frac{1}{3}\lambda x_1^{1/2}x_2^{-2/3}$	$\dfrac{\partial\mathcal{L}(x_1,x_2,\lambda)}{\partial \lambda} = -x_1^{1/2}x_2^{1/3} + 33$
$\frac{\partial^2\mathcal{L}}{\partial x_1\partial x_1} = \frac{1}{4}\lambda x_2^{1/3}x_1^{-3/2}$	$\frac{\partial^2\mathcal{L}}{\partial x_1\partial x_2} = -\frac{\lambda}{6}x_2^{-2/3}x_1^{-1/2}$	-$\frac{\partial^2\mathcal{L}}{\partial x_1\partial \lambda} = \frac{1}{2}x_2^{1/3}x_1^{-1/2}$
$\frac{\partial^2\mathcal{L}}{\partial x_2\partial x_1} = -\frac{\lambda}{6}x_1^{-1/2}x_2^{-2/3}$	$\frac{\partial^2\mathcal{L}}{\partial x_2\partial x_2} = \frac{2}{9}\lambda x_2^{-5/3}x_1^{1/2}$	- $\frac{\partial^2\mathcal{L}}{\partial x_2\partial \lambda} = \frac{1}{3}x_2^{-2/3}x_1^{1/2}$
-$\frac{\partial^2\mathcal{L}}{\partial \lambda\partial x_1} = \frac{1}{2}x_1^{-1/2}x_2^{1/3}$	-$\frac{\partial^2\mathcal{L}}{\partial \lambda\partial x_2} = \frac{1}{3}x_1^{1/2}x_2^{-2/3}$	-$\frac{\partial^2\mathcal{L}}{\partial \lambda\partial \lambda} = 0$

d. $\mathcal{L}(x_1, x_2, \lambda) \ = \ x_1^{1/3} x_2^{2/7} - \lambda\,(7x_1 + 3x_2 - 832)$

$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = \frac{1}{3} x_1^{-2/3} x_2^{2/7} - 7\lambda$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = \frac{2}{7} x_1^{1/3} x_2^{-5/7} - 3\lambda$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = -7x_1 - 3x_2 + 832$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -\dfrac{2x_2^{2/7}}{9x_1^{5/3}}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = \dfrac{2}{21} x_1^{-2/3} x_2^{-5/7}$	-$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = 7$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = \dfrac{2}{21} x_1^{-2/3} x_2^{-5/7}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -\frac{10}{49} x_1^{1/3} x_2^{-12/7}$	- $\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = 3$
- $\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1} = 7$	-$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} = 3$	-$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0$