ECONOMICS 207
SPRING 2008
LABORATORY EXERCISE 14

For your information, the Hessian matrix in the profit maximization problem written as

$$\pi(x_1,\, x_2) \;=\; pf(x_1,\, x_2) \;- w_1 x_1 - w_2 x_2$$

is given by

$$H(\pi(x_1,\, x_2)) = \begin{bmatrix} \frac{\partial^2 \pi(x_1, x_2)}{\partial x_1 \partial x_1} & \frac{\partial^2 \pi(x_1, x_2)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \pi(x_1, x_2)}{\partial x_2 \partial x_1} & \frac{\partial^2 \pi(x_1, x_2)}{\partial x_2 \partial x_2} \end{bmatrix}$$

The bordered Hessian in the constrained optimization problem written as

$$\mathcal{L}(x_1, x_2, \lambda) \;=\; f(x_1, x_2) - \lambda g(x_1, x_2)$$

is given by

$$H_B = \begin{bmatrix} \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1 \partial x_1} & \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1 \partial x_2} & -\frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1 \partial \lambda} \\ \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2 \partial x_1} & \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2 \partial x_2} & -\frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2 \partial \lambda} \\ -\frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda \partial x_1} & -\frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda \partial x_2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1 \partial x_1} & \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1 \partial x_2} & \frac{\partial g(x_1, x_2)}{\partial x_1} \\ \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2 \partial x_1} & \frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2 \partial x_2} & \frac{\partial g(x_1, x_2)}{\partial x_2} \\ \frac{\partial g(x_1, x_2)}{\partial x_1} & \frac{\partial g(x_1, x_2)}{\partial x_2} & 0 \end{bmatrix}$$

where we use the equivalencies

$$\frac{\partial g(x_1, x_2)}{\partial x_1} = -\frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1 \partial \lambda}$$

$$\frac{\partial g(x_1, x_2)}{\partial x_2} = -\frac{\partial^2 \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2 \partial \lambda}.$$

Date: April 23, 2008.

Problem 1. Given the data below, write an equation that represents profit as a function of the two inputs x_1 and x_2. Write it in the form $\pi = pf(x_1,x_2)-w_1x_1-w_2x_2$ and then simplify the expression. Then find all first and second partial derivatives of the function.

a.

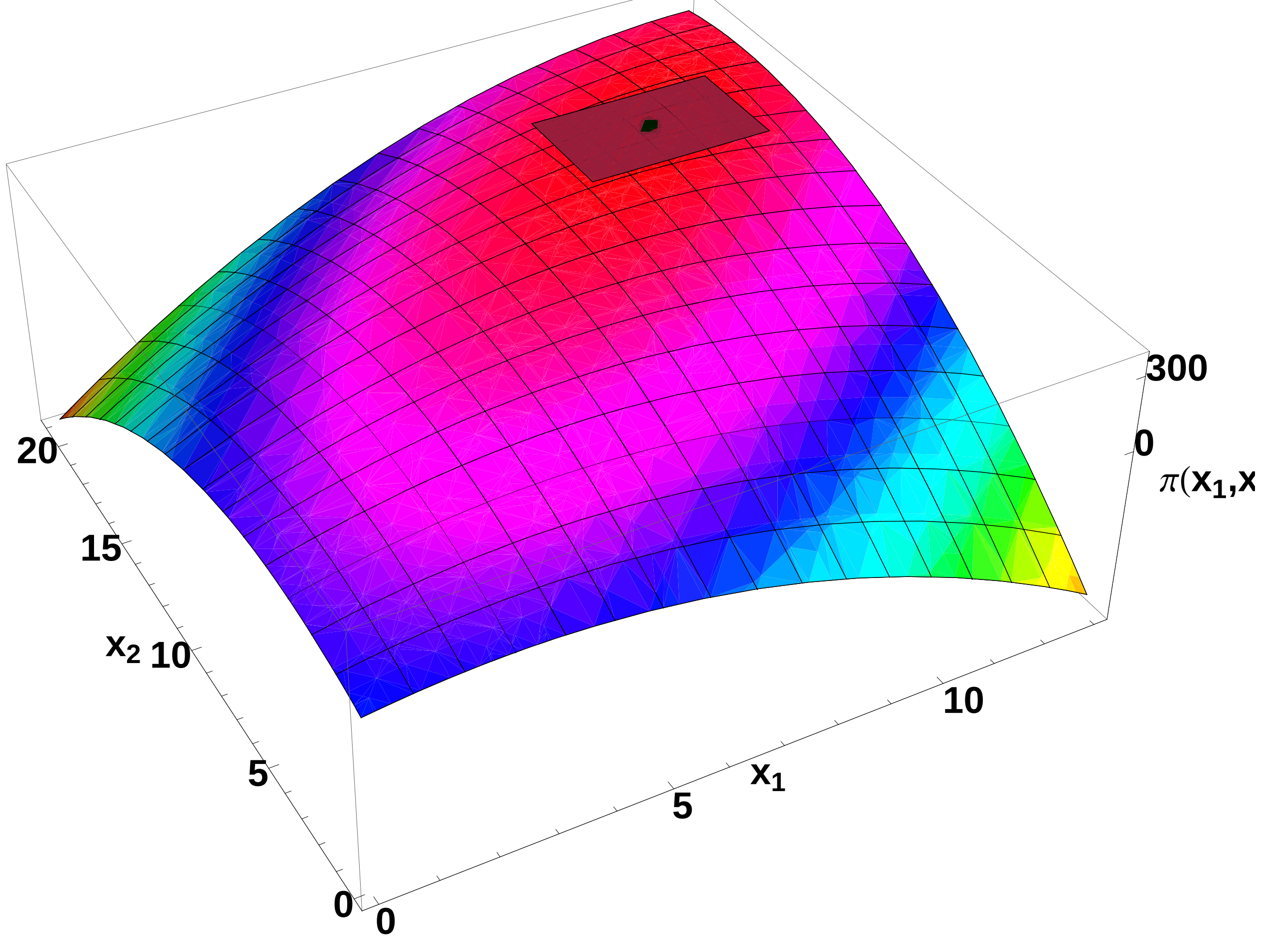
$$
\begin{aligned}
f(x_1, x_2) &= 50x_1 + 40x_2 - 2x_1^2 + 2x_1x_2 - x_2^2 \\
p &= 3 \\
w_1 &= 120, \qquad w_2 \;=\; 90
\end{aligned}
$$

$\pi =$

$\frac{\partial \pi}{\partial x_1}$	$\frac{\partial \pi}{\partial x_2}$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2}$

Find potential profit maximizing levels of x_1 and x_2.

Figure 1. Maximum Profit



By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

$$\begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1 \partial x_1} = & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} = \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} = & \frac{\partial^2 \pi}{\partial x_2 \partial x_2} = \end{vmatrix} =$$

Problem 2. a. Given the data below, write an equation that represents profit as a function of the two inputs x_1 and x_2. Write it in the form π = $pf(x_1,x_2)-w_1x_1-w_2x_2$ and then simplify the expression. Then find all first and second partial derivatives of the function.

$$f(x_1, x_2) = x_1^{2/5} x_2^{1/3}$$

$$p = 540$$

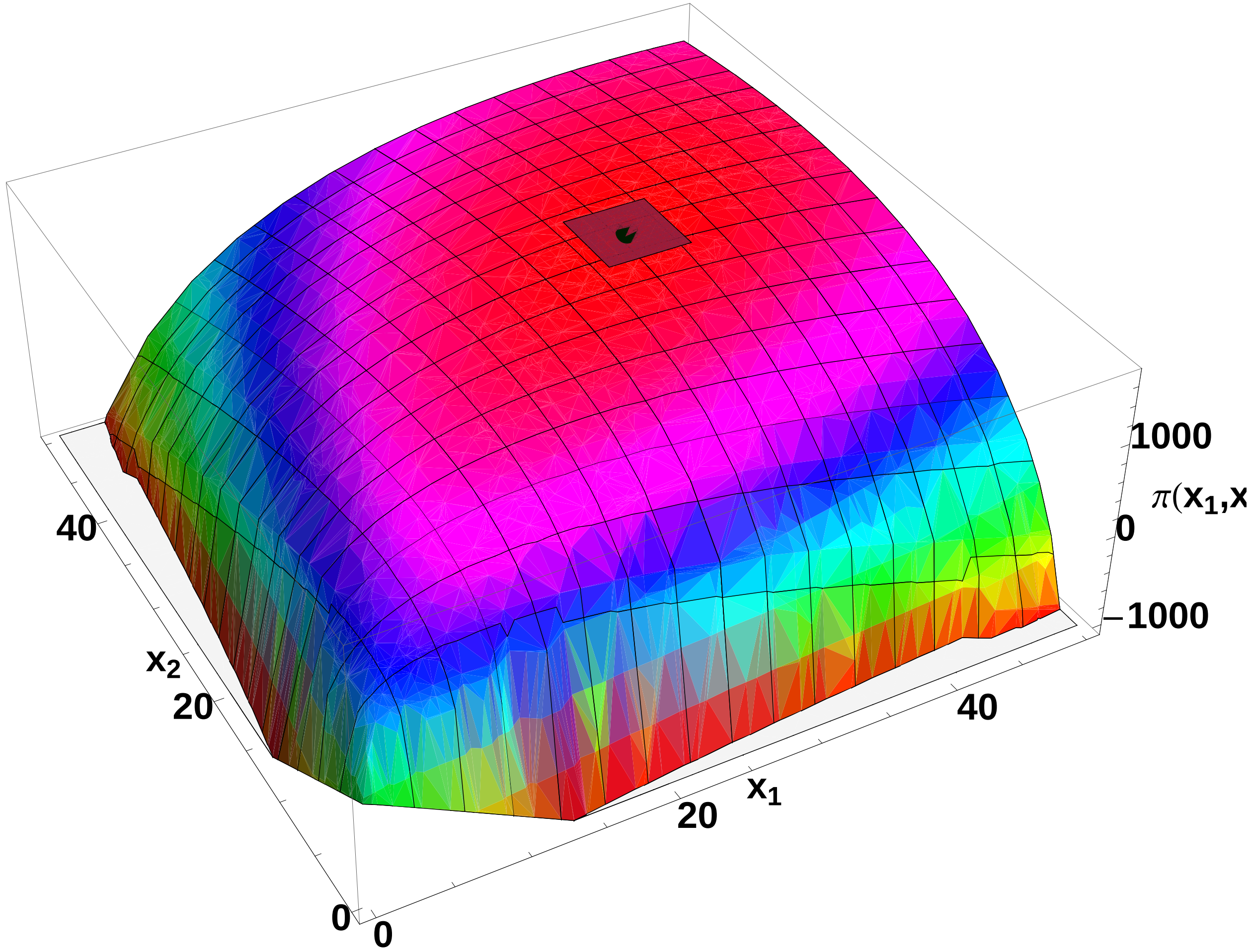
$$w_1 = 81, \qquad w_2 = 80$$

$$\pi =$$

$\frac{\partial \pi}{\partial x_1} =$	$\frac{\partial \pi}{\partial x_2} =$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} =$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} =$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} =$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} =$

b. Show that the profit maximizing levels of x_1 and x_2 are 32 and 27.

Figure 2. Maximum Profit



c. In this table fill in values of x_1 and x_2 given to obtain numerical answers for the Hessian matrix.

$\frac{\partial \pi}{\partial x_1} =$	$\frac{\partial \pi}{\partial x_2} =$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} =$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} =$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} =$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} =$

d. By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

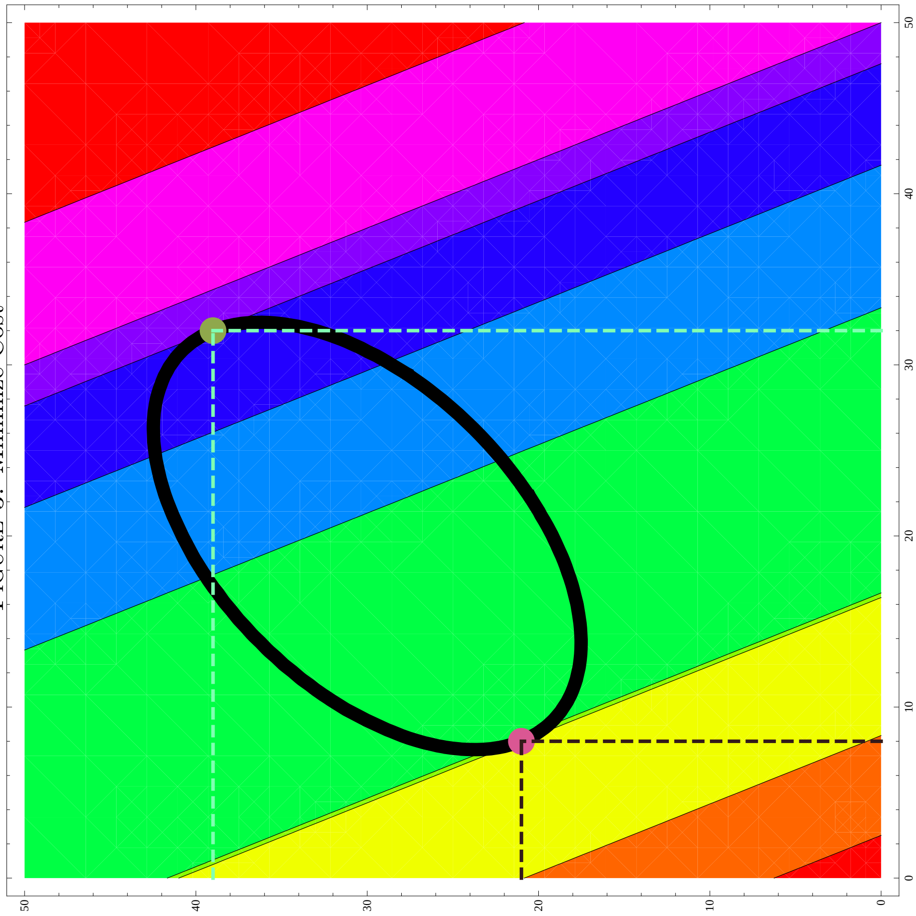
$$\begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1 \partial x_1} = & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} = \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} = & \frac{\partial^2 \pi}{\partial x_2 \partial x_2} = \end{vmatrix} =$$

Problem 3. a. Find the listed partial derivatives of following function.

$$\mathcal{L}(x_1, x_2, \lambda) \ = \ 60x_1 + 24x_2 - \lambda \left(10x_1 + 40x_2 - x_1^2 + x_1 x_2 - x_2^2 - 583\right)$$

$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 60 - \lambda(10 - 2x_1 + x_2)$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} =$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} =$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = 2\lambda$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} =$	$\frac{\partial g(x_1, x_2)}{\partial x_1} = 10 - 2x_1 + x_2$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} =$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} =$	$- \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} =$
$\frac{\partial g(x_1, x_2)}{\partial x_1} =$	$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2} =$	$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0$

Figure 3. Minimize Cost



b. This problem has two sets of roots as can be seen in figure 3. One set will maximize cost subject to the output or utility constraint, the other will minimize cost subject to the output or utility constraint, Show that three critical values of the function $\mathcal{L}(x_1, x_2, \lambda)$ are $x_1 = 32$, $x_2 = 39$, and λ = -4. Then show that the other set of roots is $x_1 = 8$, $x_2 = 21$, and $\lambda = 4$.

The first order conditions for this problem are

$$60 - \lambda(10 - 2x_1 + x_2) = 0 \tag{3.1a}$$

$$24 - \lambda(40 + x_1 - 2x_2) = 0 \tag{3.1b}$$

$$-\left(10x_1 + 40x_2 - x_1^2 + x_1 x_2 - x_2^2 - 583\right) = 0 \tag{3.1c}$$

We can rewrite this as

$$60 = \lambda(10 - 2x_1 + x_2) \tag{3.2a}$$

$$24 = \lambda(40 + x_1 - 2x_2) \tag{3.2b}$$

$$10x_1 + 40x_2 - x_1^2 + x_1 x_2 - x_2^2 = 583 \tag{3.2c}$$

We can divide equation 3.2a by equation 3.2b to eliminate λ as follows.

$$\begin{aligned}
\frac{\lambda(10 - 2x_1 + x_2)}{\lambda(40 + x_1 - 2x_2)} &= \frac{60}{24} = \frac{15}{6} \\
\Rightarrow 6(10 - 2x_1 + x_2) &= 15(40 + x_1 - 2x_2) \\
\Rightarrow 60 - 12x_1 + 6x_2 &= 600 + 15x_1 - 30x_2 \\
\Rightarrow 36x_2 &= 540 + 27x_1 \\
\Rightarrow x_2 &= \frac{540 + 27x_1}{36} \\
&= 15 + \frac{3}{4}x_1
\end{aligned} \tag{3.3}$$

More space for Problem 3.

Now we substitute for x_2 in equation 3.2c to get an equation in x_1 only.

$$
\begin{aligned}
10x_1 + 40x_2 - x_1^2 + x_1x_2 - x_2^2 =& 583 \\
\Rightarrow 10x_1 + 40\left(15 + \frac{3}{4}x_1\right) - x_1^2 + x_1\left(15 + \frac{3}{4}x_1\right) - \left(15 + \frac{3}{4}x_1\right)^2 =& 583 \\
\Rightarrow 10x_1 + 600 + 30x_1 - x_1^2 + 15x_1 + \frac{3}{4}x_1^2 - \left(225 + \frac{45}{2}x_1 + \frac{9}{16}x_1^2\right) =& 583 \\
\Rightarrow 10x_1 + 30x_1 + 15x_1 - \frac{45}{2}x_1 - x_1^2 + \frac{3}{4}x_1^2 - \frac{9}{16}x_1^2 + 600 - 225 - 583 =& 0
\end{aligned}
\tag{3.4}
$$

Now collect terms in x_1^2 and x_1 to simplify the expression.

$$
\begin{aligned}
10x_1 + 30x_1 + 15x_1 - \frac{45}{2}x_1 - x_1^2 + \frac{3}{4}x_1^2 - \frac{9}{16}x_1^2 + 600 - 225 - 583 =& 0 \\
\Rightarrow (-1 + \frac{3}{4} - \frac{9}{16})x_1^2 + (10 + 30 + 15 - \frac{45}{2})x_1 - 208 =& 0 \\
\Rightarrow \frac{-13}{16}x_1^2 + \frac{65}{2}x_1 - 208 =& 0
\end{aligned}
\tag{3.5}
$$

Now multiply both sides of the last expression in equation 3.5 by 16 to obtain

$$
\begin{aligned}
\frac{-13}{16}x_1^2 + \frac{65}{2}x_1 - 208 =& 0 \\
\Rightarrow -13x_1^2 + (8 \times 65)x_1 - (208)(16) =& 0 \\
\Rightarrow -13x_1^2 + (2^3 \times 5 \times 13)x_1 - (2^4 \times 13)(2^4) =& 0 \\
\Rightarrow -x_1^2 + (2^3 \times 5)x_1 - (2^4)(2^4) =& 0 \\
\Rightarrow x_1^2 - 40x_1 + 256 =& 0 \\
\Rightarrow (x_1 - 8)(x_1 - 32) =& 0 \\
\Rightarrow x_1 = 8 \quad \text{or} \quad x_1 =& 32
\end{aligned}
\tag{3.6}
$$

Now substitute for x_1 in equation 3.3

$$
\begin{aligned}
x_2 =& 15 + \frac{3}{4} x_1 \\
=& 15 + \frac{3}{4}(8) = 21 \\
& \text{or} \\
x_2 =& 15 + \frac{3}{4}(32) = 39
\end{aligned}
\tag{3.7}
$$

There will be two values od λ, one for each set of x values. We can find them using either equation 3.2a or 3.2b.

$$
\begin{aligned}
60 =& \lambda(10 - 2x_1 + x_2) \\
\Rightarrow \lambda =& \frac{60}{10 - 2x_1 + x_2}
\end{aligned}
\tag{3.8}
$$

For $x_1 = 8$ and $x_2 = 21$ we obtain

$$
\begin{aligned}
\lambda =& \frac{60}{10 - 2x_1 + x_2} \\
=& \frac{60}{10 - 2(8) + 21} \\
=& \frac{60}{15} = 4
\end{aligned}
\tag{3.9}
$$

Similary for $x_1 = 32$ and $x_2 = 39$ we obtain

$$
\begin{aligned}
\lambda =& \frac{60}{10 - 2x_1 + x_2} \\
=& \frac{60}{10 - 2(32) + 39} \\
=& \frac{60}{10 - 64 + 39} \\
=& \frac{60}{-15} = -4
\end{aligned}
\tag{3.10}
$$

c. By substituting $x_1 = 32$, $x_2 = 39$ and $\lambda = -4$ into the bordered Hessian matrix, show that this point maximizes the cost of producing 583 units of output Hint: The determinant with be 2808.

$$\left| \begin{array}{ccc} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -8 & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = 4 & -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = -15 \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = \\ \frac{\partial g(x_1, x_2)}{\partial x_1} = & \frac{\partial g(x_1, x_2)}{\partial x_2} = & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = \end{array} \right| =$$

A positive determinant indicates a maximum, a negative determinant indicates a minimum.

d. By substituting $x_1 = 8$, $x_2 = 21$ and $\lambda = 4$ into the bordered Hessian matrix, show that this point minimizes the cost of producing 583 units of output Hint: The determinant with be -2808.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = 8 & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = -4 & \frac{\partial g(x_1,x_2)}{\partial x_1} = \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = \\ \frac{\partial g(x_1,x_2)}{\partial x_1} = & \frac{\partial g(x_1,x_2)}{\partial x_2} = & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = \end{vmatrix} =$$

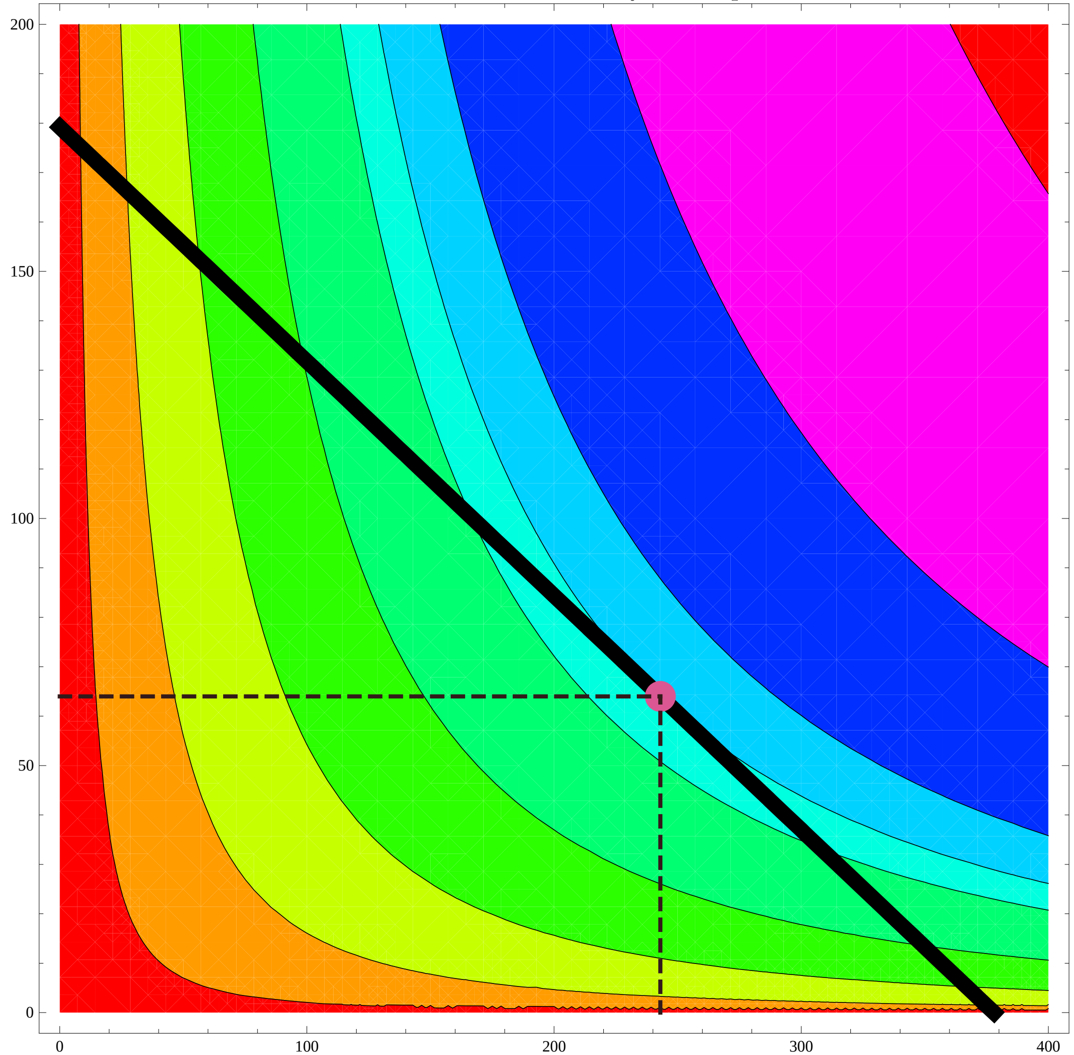
A positive determinant indicates a maximum, a negative determinant indicates a minimum.

Problem 4. a. Find the listed partial derivatives of following function.

$$\mathcal{L}(x_1, x_2, \lambda) \; = \; x_1^{3/5} x_2^{1/3} - \lambda \left(64x_1 + 135x_2 - 24192\right)$$

$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = \frac{3}{5} x_1^{-2/5} x_2^{1/3} - 64\lambda$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = \frac{1}{3} x_1^{3/5} x_2^{-2/3} - 135\lambda$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = -\left(64x_1 + 135x_2 - 24192\right)$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -\frac{6}{25} x_1^{-7/5} x_2^{1/3}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} =$	$\frac{\partial g(x_1, x_2)}{\partial x_1} = 64$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} =$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -\frac{2x_1^{3/5}}{9x_2^{5/3}}$	$\frac{\partial g(x_1, x_2)}{\partial x_2} = 135$
- $\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1}$	-$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2}$	$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} =$

Figure 4. Maximize Utility or Output



b. Show that the three critical values of the function $\mathcal{L}(x_1, x_2, \lambda)$ are $x_1 = 243$, $x_2 = 64$, and $\lambda = \frac{1}{240}$.

Hints:

(a) $64 = 2^6$

(b) $135 = 5 \times 27 = 5 \times 3^3$

(c) $24192 = 2^7 \times 3^3 \times 7$

To solve take the ratio of the first two first order conditions and solve for x_2 as a function of x_1 and then substitute into the third first order condition.

$$
\begin{aligned}
\frac{\frac{3}{5}x_1^{-2/5}x_2^{1/3}}{\frac{1}{3}x_1^{3/5}x_2^{-2/3}} &= \frac{64\lambda}{135\lambda} \\
\Rightarrow \frac{9}{5}x_1^{-1}x_2 &= \frac{64}{135} \\
\Rightarrow x_1^{-1}x_2 &= \frac{5 \times 64}{9 \times 135} \\
\Rightarrow x_2 &= \frac{64}{9 \times 27}x_1 \\
\Rightarrow x_2 &= \frac{2^6}{3^5}x_1 \\
\Rightarrow x_2 &= 2^6 3^{-5} x_1
\end{aligned}
$$

Now substitute

$$
\begin{aligned}
64x_1 + 135x_2 &= 24192 = 2^7 \times 3^3 \times 7 \\
\Rightarrow 64x_1 + 135 \times 2^6 3^{-5} x_1 &= 2^7 \times 3^3 \times 7 \\
\Rightarrow 2^6 x_1 + 5 \times 3^3 2^6 3^{-5} x_1 &= 2^7 \times 3^3 \times 7 \\
\Rightarrow 2^6 x_1 (1 + 5 \times 3^3 3^{-5}) &= 2^7 \times 3^3 \times 7 \\
\Rightarrow x_1 &= \frac{2^7 \times 3^3 \times 7}{2^6 \times (1 + 5 \times 3^{-2})} \\
&= \frac{2 \times 3^3 \times 7}{(1 + \frac{5}{9})} = \frac{2 \times 3^3 \times 7}{\frac{14}{9}} = \frac{2 \times 3^5 \times 7}{14} = 3^5 = 243
\end{aligned}
$$

c. Substitute the appropriate values of are x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is 56. Hint: $25 \times 729 = 18225$.

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -\frac{8}{18225} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = \frac{1}{720} & \frac{\partial g(x_1,x_2)}{\partial x_1} = 64 \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = \\ \frac{\partial g(x_1,x_2)}{\partial x_1} = & \frac{\partial g(x_1,x_2)}{\partial x_2} = & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0 \end{vmatrix} =$$

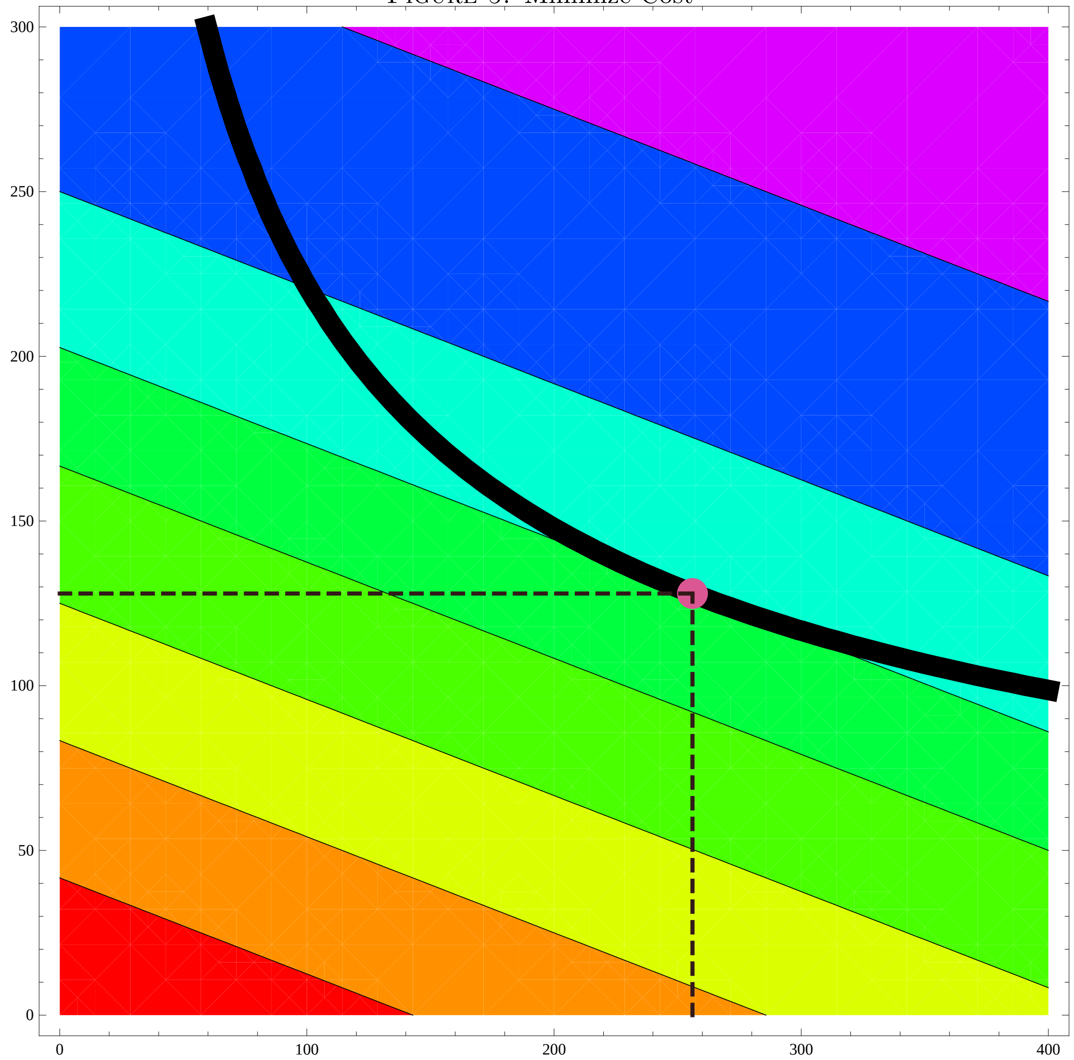
A positive determinant indicates a maximum, a negative determinant indicates a minimum.

Problem 5. a. Find the listed partial derivatives of following function.

$$\mathcal{L}(x_1, x_2, \lambda) \ = \ 7x_1 + 24x_2 - \lambda \left(x_1^{1/4} x_2^{3/7} - 32 \right)$$

$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 7 - \frac{1}{4}\lambda x_1^{-3/4} x_2^{3/7}$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = 24 - \frac{3}{7}\lambda x_1^{1/4} x_2^{-4/7}$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} =$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} =$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = -\frac{3\lambda}{28 x_1^{3/4} x_2^{4/7}}$	$\frac{\partial g(x_1, x_2)}{\partial x_1} =$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} =$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} =$	$\frac{\partial g(x_1, x_2)}{\partial x_2} =$
- $\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1}$	-$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2}$	$\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0$

Figure 5. Minimize Cost



b. Show that the three critical values of the function $\mathcal{L}(x_1, x_2, \lambda)$ are $x_1 = 256$, $x_2 = 128$, and $\lambda = 224$.

c. Substitute the appropriate values of are x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is $-\frac{57}{114688} = \frac{57}{2^{14}\times 7}$. Hints: $2^{10} = 1024; 128 \times 32 \times 28 = 2^7 \times 2^5 \times 2^2 \times 7 = 2^{14} \times 7 = 114688$

$$\begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = \frac{21}{1024} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = -\frac{3}{128} & -\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} = \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = & -\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} = \\ \frac{\partial g(x_1,x_2)}{\partial x_1} = & \frac{\partial g(x_1,x_2)}{\partial x_2} = \frac{3}{28} & \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda} = 0 \end{vmatrix} =$$

A positive determinant indicates a maximum, a negative determinant indicates a minimum.