ECONOMICS 207
SPRING 2008
LABORATORY EXERCISE 15
KEY

For your information, the Hessian matrix in the profit maximization problem written as

$$\pi(x_1,\, x_2) \;=\; pf(x_1,\, x_2) \;-w_1x_1 - w_2x_2$$

is given by

$$H(\pi(x_1,\, x_2)) = \begin{bmatrix} \dfrac{\partial^2\pi(x_1,x_2)}{\partial x_1\partial x_1} & \dfrac{\partial^2\pi(x_1,x_2)}{\partial x_1\partial x_2} \\ \dfrac{\partial^2\pi(x_1,x_2)}{\partial x_2\partial x_1} & \dfrac{\partial^2\pi(x_1,x_2)}{\partial x_2\partial x_2} \end{bmatrix}$$

The bordered Hessian in the constrained optimization problem written as

$$\mathcal{L}(x_1,x_2,\lambda) \;=\; f(x_1,x_2) - \lambda g(x_1,x_2)$$

is given by

$$H_B = \begin{bmatrix} \dfrac{\partial^2\mathcal{L}(x_1,x_2,\lambda)}{\partial x_1\partial x_1} & \dfrac{\partial^2\mathcal{L}(x_1,x_2,\lambda)}{\partial x_1\partial x_2} & -\dfrac{\partial^2\mathcal{L}(x_1,x_2,\lambda)}{\partial x_1\partial \lambda} \\ \dfrac{\partial^2\mathcal{L}(x_1,x_2,\lambda)}{\partial x_2\partial x_1} & \dfrac{\partial^2\mathcal{L}(x_1,x_2,\lambda)}{\partial x_2\partial x_2} & -\dfrac{\partial^2\mathcal{L}(x_1,x_2,\lambda)}{\partial x_2\partial \lambda} \\ -\dfrac{\partial^2\mathcal{L}(x_1,x_2,\lambda)}{\partial \lambda\partial x_1} & -\dfrac{\partial^2\mathcal{L}(x_1,x_2,\lambda)}{\partial \lambda\partial x_2} & 0 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial^2\mathcal{L}(x_1,x_2,\lambda)}{\partial x_1\partial x_1} & \dfrac{\partial^2\mathcal{L}(x_1,x_2,\lambda)}{\partial x_1\partial x_2} & \dfrac{\partial g(x_1,x_2)}{\partial x_1} \\ \dfrac{\partial^2\mathcal{L}(x_1,x_2,\lambda)}{\partial x_2\partial x_1} & \dfrac{\partial^2\mathcal{L}(x_1,x_2,\lambda)}{\partial x_2\partial x_2} & \dfrac{\partial g(x_1,x_2)}{\partial x_2} \\ \dfrac{\partial g(x_1,x_2)}{\partial x_1} & \dfrac{\partial g(x_1,x_2)}{\partial x_2} & 0 \end{bmatrix}$$

where we use the equivalencies

$$\frac{\partial g(x_1,x_2)}{\partial x_1} = -\frac{\partial^2\mathcal{L}(x_1,x_2,\lambda)}{\partial x_1\partial\lambda}$$

$$\frac{\partial g(x_1,x_2)}{\partial x_2} = -\frac{\partial^2\mathcal{L}(x_1,x_2,\lambda)}{\partial x_2\partial\lambda}.$$

Date: August 5, 2008.

Problem 1. Below you are given a production function for a competitive firm. You are also given the price of the firm's output and the prices of the two inputs used by the firm. Output price is represented by p, the price of the first input by w_1 and the price of the second input by w_2.

Hint: 560 - 144 = 416.

$$f(x_1,x_2) = 40x_1 + 20x_2 - 2x_1^2 + x_1x_2 - x_2^2$$
$$p = 10$$
$$w_1 = 120, \qquad w_2 = 60$$

a. Write an equation that represents profit as a function of the two inputs x_1 and x_2. Simplify the expression.

$$\begin{aligned} Profit &= pf(x_1,x_2) - w_1x_1 - w_2x_2 \\ &= 10(40x_1 + 20x_2 - 2x_1^2 + x_1x_2 - x_2^2) - 120x_1 - 60x_2 \\ &= 280x_1 + 140x_2 - 20x_1^2 + 10x_1x_2 - 10x_2^2 \end{aligned} \quad (1)$$

b. Find all first and second partial derivatives of the function.

$\frac{\partial \pi}{\partial x_1} = 280 - 40x_1 + 10x_2$	$\frac{\partial \pi}{\partial x_2} = 140 + 10x_1 - 20x_2$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -40$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 10$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 10$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -20$

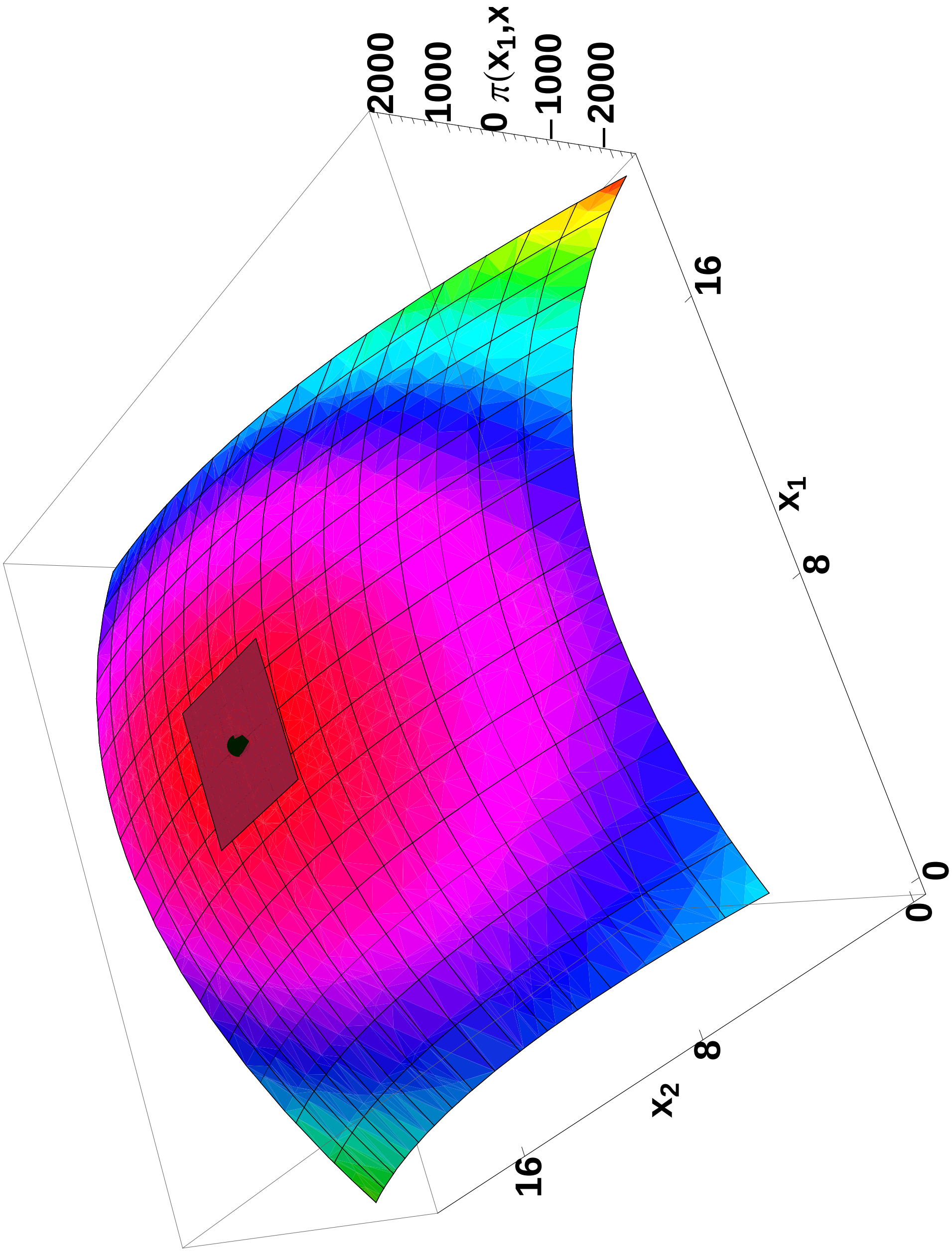


FIGURE 1. Maximum Profit

c. Find potential profit maximizing levels of x_1 and x_2.

By setting the first derivative of equation (1) to zero, we obtain

$$280 - 40x_1 + 10x_2 = 0 \tag{2}$$

$$140 + 10x_1 - 20x_2 = 0 \tag{3}$$

Simplify equation (2) and (3).

$$28 - 4x_1 + x_2 = 0 \tag{4}$$

$$14 + x_1 - 2x_2 = 0 \tag{5}$$

Add equation (5) multiplied by 4 to equation (4).

$$\begin{aligned} & 28 - 4x_1 + x_2 + 4(14 + x_1 - 2x_2) &=& \ 0 \\ \Rightarrow & 84 - 7x_2 &=& \ 0 \\ \Rightarrow & x_2 &=& \ 12 \end{aligned}$$

Substitute $x_2 = 12$ into equation (4).

$$\begin{aligned} & 28 - 4x_1 + x_2 &=& \ 0 \\ \Rightarrow & 28 - 4x_1 + 12 &=& \ 0 \\ \Rightarrow & 40 - 4x_1 &=& \ 0 \\ \Rightarrow & x_1 &=& \ 10 \end{aligned}$$

So the potential profit maximizing levels of x_1 and x_2 is

$$x_1 = 10,\ x_2 = 12$$

d. Fill in the elements of the Hessian matrix of the profit equation evaluated at the critical values of x_1 and x_2 and then verify that the levels of x_1 and x_2 you found are either maximum, minimum or saddle points.

$$H = \begin{vmatrix} \frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -40 & \frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 10 \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 10 & \frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -20 \end{vmatrix} = 800 - 100 = 700 > 0$$

Both diagonal elements are negative and the determinant of the Hessian is positive, so the input levels $x_1 = 10$, $x_2 = 12$ represent a point of profit maximization.

e. What is the optimal level of output?

When $x_1 = 10$, $x_2 = 12$, the output is given by

$$\begin{aligned} f(10,12) &= 40 \times 10 + 20 \times 12 - 2 \times 10^2 + 10 \times 12 - 12^2 \\ &= 400 + 240 - 200 + 120 - 144 \\ &= 416 \end{aligned}$$

f. How much does the firm spend on inputs?

When $x_1 = 10$, $x_2 = 12$, the cost of inputs is given by

$$w_1 x_1 + w_2 x_2 = 120 \times 10 + 60 \times 12 = 1920$$

Problem 2. Consider a consumer with a utility function given by

$$v = u(x_1, x_2) = 40x_1 + 20x_2 - 2x_1^2 + x_1 x_2 - x_2^2$$

The consumer faces prices and an income constraint given by

$$p_1 = 120, \qquad p_2 = 60, \qquad m_0 = 1920$$

Find potential levels of x_1 and x_2 to maximum utility for this consumer given the income constraint and the stated prices. Verify that these consumption levels maximize utility.

Hint: 1920/160=12; $4 \times 6 \times 6 = 2^4 \times 3^2 = 144$.

a. Set up the objective function for this problem and find all first and second partial derivatives of the function with respect to x_1 and x_2.

$$\mathcal{L}(x_1, x_2, \lambda) = 40x_1 + 20x_2 - 2x_1^2 + x_1 x_2 - x_2^2 - \lambda(120x_1 + 60x_2 - 1920) \tag{6}$$

$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 40 - 4x_1 + x_2 - 120\lambda$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = 20 + x_1 - 2x_2 - 60\lambda$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -4$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = 1$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = 1$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -2$

b. What is the derivative of the objective function in this problem with respect to λ?

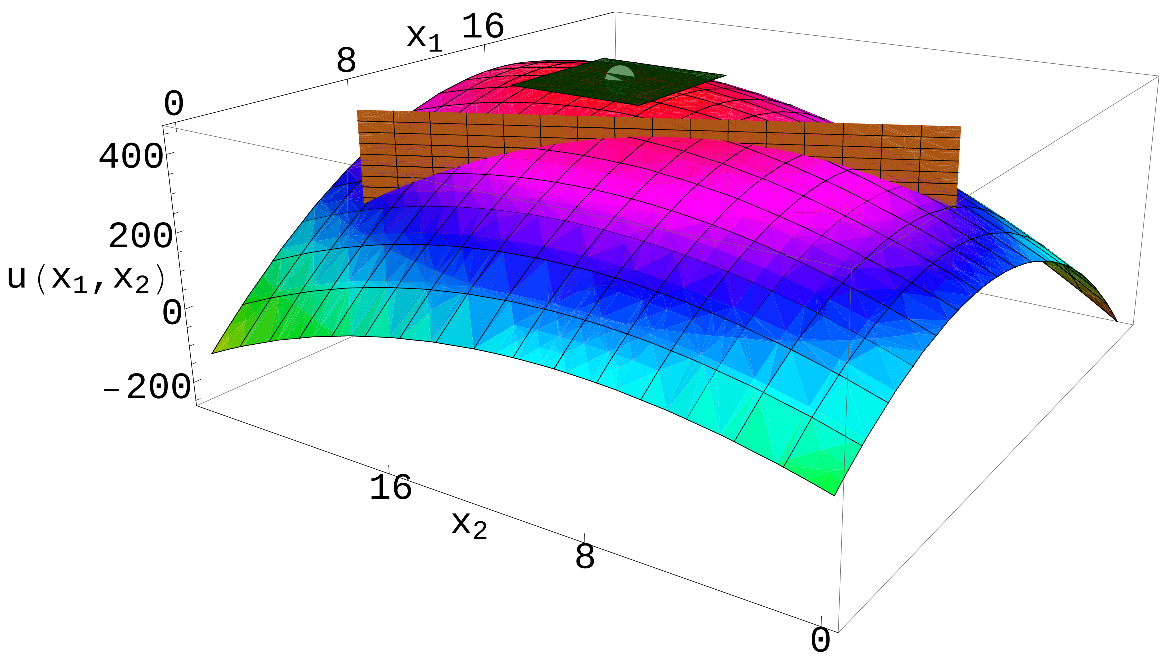
$$\begin{aligned}\frac{d\,\mathcal{L}(x_1, x_2, \lambda)}{d\,\lambda} &= \frac{d\,\left(40x_1 + 20x_2 - 2x_1^2 + x_1x_2 - x_2^2 - \lambda(120x_1 + 60x_2 - 1920)\right)}{d\,\lambda} \\ &= -(120x_1 + 60x_2 - 1920)\end{aligned}$$

c. Find the partial derivatives of the constraint equation with respect to x_1 and x_2.

$\frac{\partial g(x_1, x_2)}{\partial x_1} = 120$	$\frac{\partial g(x_1, x_2)}{\partial x_2} = 60$

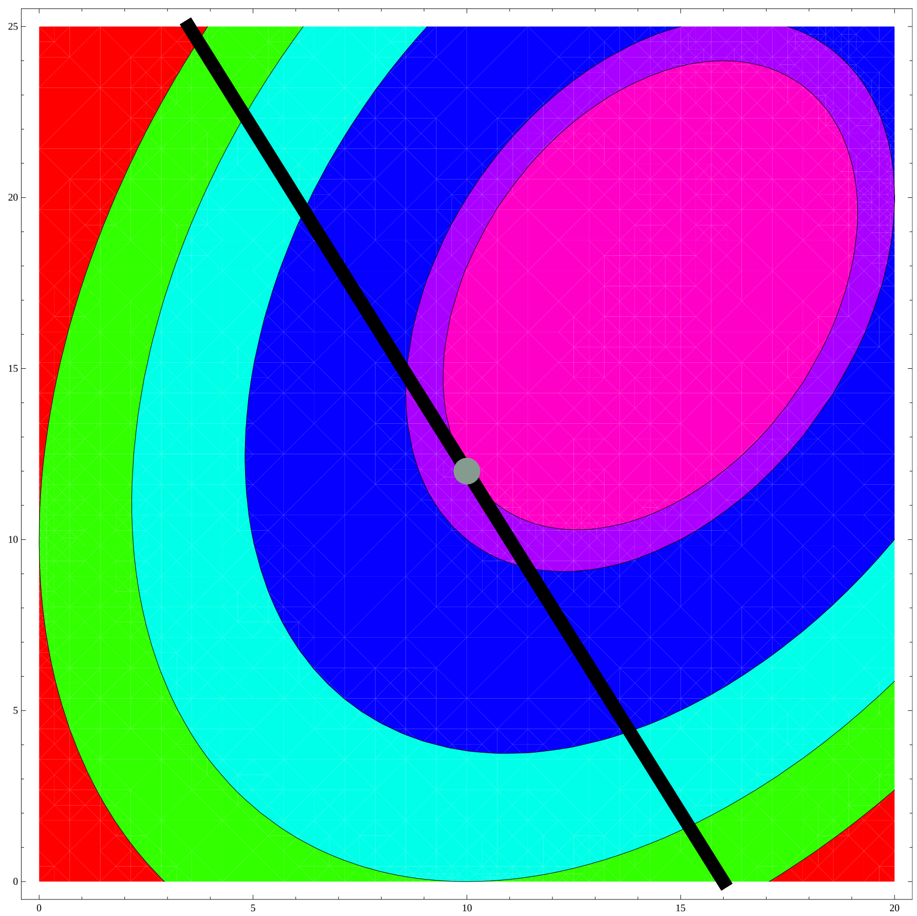
In figure 2 you can see the maximum utility point and how it is unattainable given the budget constraint.

FIGURE 2. Utility and the Budget Constraint



In figure 3 you can see the tangency between one indifference curve and the budget line.

FIGURE 3. Tangency Between Indifference Curve and Budget Line



d. Use the information from 2a and 2b to find critical values for x_1, x_2 and λ.

Setting the first derivative to zero, we obtain

$$40 - 4x_1 + x_2 - 120\lambda = 0 \tag{7}$$

$$20 + x_1 - 2x_2 - 60\lambda = 0 \tag{8}$$

$$120x_1 + 60x_2 - 1920 = 0 \tag{9}$$

Subtract equation (8) multiplied by 2 from equation (7).

$$\begin{aligned} 40 - 4x_1 + x_2 - 120\lambda - 2(20 + x_1 - 2x_2 - 60\lambda) &= 0 \\ \Rightarrow \quad -6x_1 + 5x_2 &= 0 \\ \Rightarrow \quad x_1 &= \frac{5}{6}x_2 \end{aligned}$$

Substitute $x_1 = \frac{5}{6}x_2$ into equation (9)

$$\begin{aligned} 120x_1 + 60x_2 - 1920 &= 0 \\ \Rightarrow \quad 120\left(\frac{5}{6}x_2\right) + 60x_2 - 1920 &= 0 \\ \Rightarrow \quad 160x_2 - 1920 &= 0 \\ \Rightarrow \quad x_2 &= 1920/160 = 12 \end{aligned}$$

Then $x_1 = \frac{5}{6}x_2 = \frac{5}{6} \times 12 = 10$.
Substitute $x_1 = 10$, $x_2 = 12$ into equation (7) .

$$\begin{aligned} 40 - 4x_1 + x_2 - 120\lambda &= 0 \\ \Rightarrow \quad 40 - 4 \times 10 + 12 - 120\lambda &= 0 \\ \Rightarrow \quad 120\lambda &= 12 \\ \Rightarrow \quad 120\lambda &= \frac{12}{120} = \frac{1}{10} \end{aligned}$$

So the critical values are

$$x_1 = 10, \; x_2 = 12, \; \lambda = \frac{1}{10}$$

e. Use the answers from part 2d and the expressions from parts 2a and 2c to fill in the bordered Hessian matrix for this problem. Then determine whether the critical values indicate a maximum, a minimum, or a saddle point. The determinant of the bordered Hessian matrix is 57,600.

$$H_B = \begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -4 & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = 1 & \frac{\partial g(x_1, x_2)}{\partial x_1} = 120 \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = 1 & \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -2 & \frac{\partial g(x_1, x_2)}{\partial x_2} = 60 \\ \frac{\partial g(x_1, x_2)}{\partial x_1} = 120 & \frac{\partial g(x_1, x_2)}{\partial x_2} = 60 & 0 \end{vmatrix}$$

$$= 120 \times \begin{vmatrix} 1 & 120 \\ -2 & 60 \end{vmatrix} - 60 \times \begin{vmatrix} -4 & 120 \\ 1 & 60 \end{vmatrix}$$

$$= 120 \times 300 + 60 \times 360 = 57600 > 0$$

So this set of critical values indicates a maximum.

f. If income went up by \$1.00, by how much would utility rise?

Let the income be m_0. Also by setting the first derivative to zero, we obtain

$$40 - 4x_1 + x_2 - 120\lambda = 0 \tag{10}$$

$$20 + x_1 - 2x_2 - 60\lambda = 0 \tag{11}$$

$$120x_1 + 60x_2 - m_0 = 0 \tag{12}$$

And similarly from equation (10) and equation (11), we can get

$$x_1 = \frac{5}{6}x_2 \tag{13}$$

Substitute $x_1 = \frac{5}{6}x_2$ into equation (12).

$$\begin{aligned} & 120x_1 + 60x_2 - m_0 &=& \ 0 \\ \Rightarrow\ & 120\left(\frac{5}{6}x_2\right) + 60x_2 - m_0 &=& \ 0 \\ \Rightarrow\ & 160x_2 - m_0 &=& \ 0 \\ \Rightarrow\ & x_2 &=& \ m_0/160 \end{aligned}$$

And then

$$\begin{aligned} x_1 &= \frac{5}{6}x_2 = \frac{5}{6}\frac{m_0}{160} \\ &= \frac{m_0}{192} \end{aligned}$$

Given the condition of $x_1 = \frac{m_0}{192}$ and $x_2 = \frac{m_0}{160}$, the utility is given by

$$\begin{aligned} u(x_1,\, x_2) &= 40x_1 + 20x_2 - 2x_1^2 + x_1x_2 - x_2^2 \\ &= \frac{40m_0}{192} + \frac{20m_0}{160} - 2\frac{m_0^2}{192^2} + \frac{m_0^2}{192 \times 160} - \frac{m_0^2}{160^2} \\ &= \frac{1}{3}m_0 - \frac{7}{115200}m_0^2 \end{aligned}$$

Then, the marginal utility is given by

$$\frac{d\,u(x_1,x_2)}{d\,m_0} = \frac{1}{3} - \frac{14}{115200}m_0$$

When $m_0 = 1920$, the marginal utility is given by

$$\frac{1}{3} - \frac{14}{115200}m_0 = \frac{1}{3} - \frac{14 \times 1920}{115200} = \frac{1}{10}$$

So, if income went up by \$1.00, the utility would rise by $\frac{1}{10}$ when the income is \$1920.

In fact, the Lagrange multiplier is the extra utility that results from an extra dollars of income. And the result of above computation is a case.

Problem 3. Below you are given a production function for a competitive firm. You are also given the price of the firm's output and the prices of the two inputs used by the firm. Output price is represented by p, the price of the first input by w_1 and the price of the second input by w_2.

$$f(x_1, x_2) = x_1^{1/5} x_2^{1/2}$$
$$p = 80$$
$$w_1 = 5, \qquad w_2 = 16$$

a. Write an equation that represents profit as a function of the two inputs x_1 and x_2. Simplify the expression.

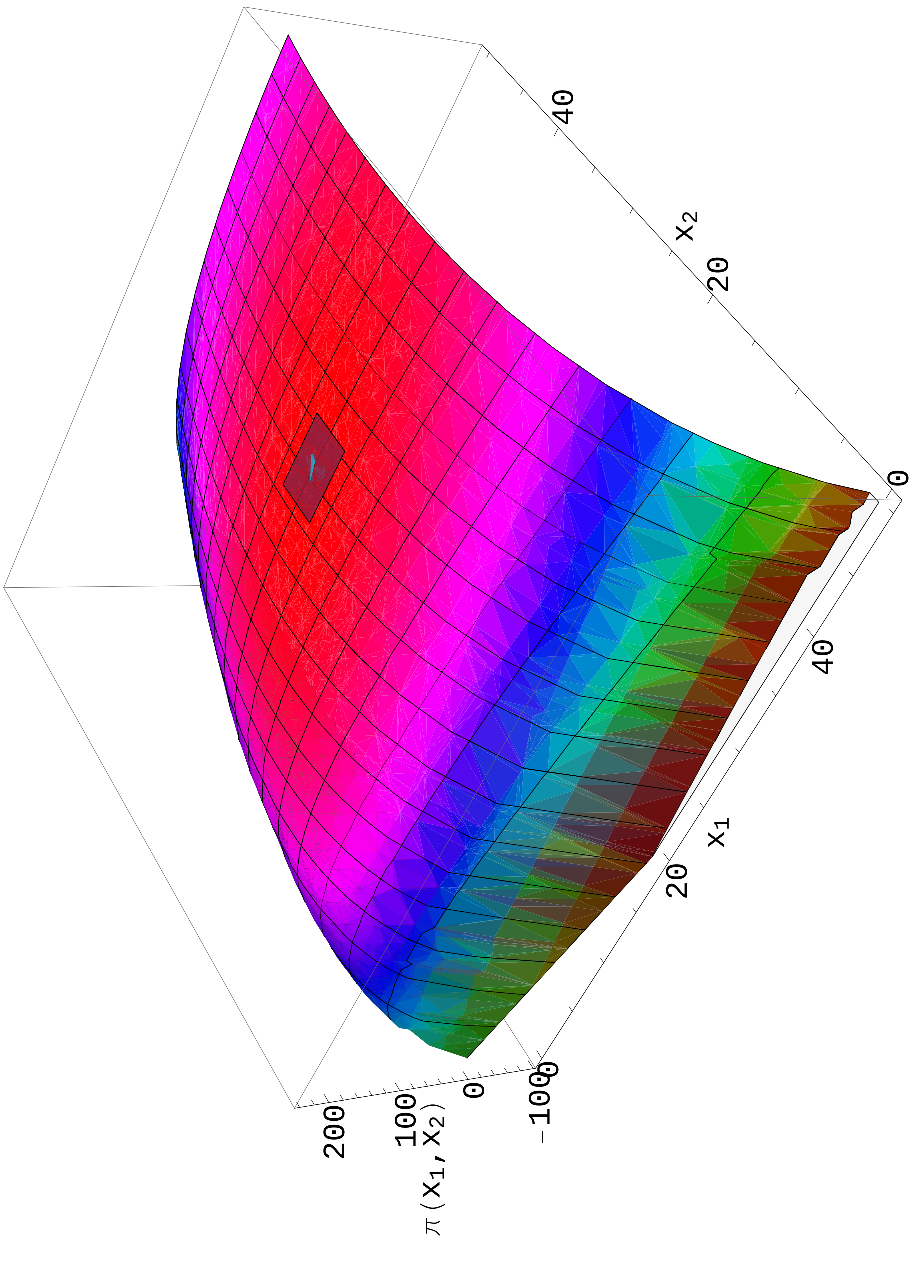
Profit is given by

$$Profit = pf(x_1, x_2) - w_1 x_1 - w_2 x_2$$
$$= 80 x_1^{1/5} x_2^{1/2} - 5x_1 - 16x_2 \qquad (14)$$

b. Find all first and second partial derivatives of the function.

$\frac{\partial \pi}{\partial x_1} = 16 x_1^{-4/5} x_2^{1/2} - 5$	$\frac{\partial \pi}{\partial x_2} = 40 x_1^{1/5} x_2^{-1/2} - 16$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -\frac{64}{5} x_1^{-9/5} x_2^{1/2}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 8 x_1^{-4/5} x_2^{-1/2}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 8 x_1^{4/5} x_2^{-1/2}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -20 x_1^{1/5} x_2^{-3/2}$

Figure 4. Maximum Profit



c. Find potential profit maximizing levels of x_1 and x_2.

By setting equation (14) to zero, we obtain

$$16x_1^{-4/5}x_2^{1/2} - 5 = 0 \tag{15}$$

$$40x_1^{1/5}x_2^{-1/2} - 16 = 0 \tag{16}$$

Rearrange equation (15), we get

$$\begin{aligned} 16x_1^{-4/5}x_2^{1/2} - 5 &= 0 \\ \Rightarrow \quad \tfrac{16}{5}x_1^{-4/5} &= x_2^{-1/2} \\ \Rightarrow \quad x_2^{-1/2} &= \tfrac{16}{5}x_1^{-4/5} \end{aligned}$$

Substitute $x_2^{-1/2} = \frac{16}{5}x_1^{-4/5}$ into equation (16).

$$\begin{aligned} 40x_1^{1/5}x_2^{-1/2} - 16 &= 0 \\ \Rightarrow \quad 40x_1^{1/5}\left(\tfrac{16}{5}x_1^{-4/5}\right) - 16 &= 0 \\ \Rightarrow \quad 40 \times \tfrac{16}{5}x_1^{-3/5} &= 16 \\ \Rightarrow \quad 8x_1^{-3/5} &= 1 \\ \Rightarrow \quad x_1^{-3/5} &= 2^{-3} \\ \Rightarrow \quad x_1^{1/5} &= 2 \\ \Rightarrow \quad x_1 &= 32 \end{aligned}$$

Then, substitute $x_1 = 32$ into $x_2^{-1/2} = \frac{16}{5}x_1^{-4/5}$.

$$\begin{aligned} x_2^{-1/2} &= \tfrac{16}{5}x_1^{-4/5} \\ \Rightarrow \quad x_2^{-1/2} &= \tfrac{16}{5}32^{-4/5} = 5^{-1} \\ \Rightarrow \quad x_2^{1/2} &= 5 \\ \Rightarrow \quad x_2 &= 25 \end{aligned}$$

d. Fill in the elements of the Hessian matrix of the profit equation evaluated at the critical values of x_1 and x_2 and then verify that the levels of x_1 and x_2 you found are either maximum, minimum or saddle points.

$$H = \begin{vmatrix} \begin{aligned} \frac{\partial^2 \pi}{\partial x_1 \partial x_1} &= -\tfrac{64}{5} x_1^{-9/5} x_2^{1/2} \\ &= -\tfrac{64}{5} 32^{-9/5} 25^{1/2} \\ &= -\tfrac{1}{8} \end{aligned} & \begin{aligned} \frac{\partial^2 \pi}{\partial x_1 \partial x_2} &= 8 x_1^{-4/5} x_2^{-1/2} \\ &= 8 \times 32^{-4/5} 25^{-1/2} \\ &= \tfrac{1}{10} \end{aligned} \\ \frac{\partial^2 \pi}{\partial x_2 \partial x_1} = \tfrac{1}{10} & \frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -\tfrac{8}{25} \end{vmatrix} = \frac{1}{25} - \frac{1}{100} = \frac{3}{100} > 0$$

The determinant of the Hessian matrix is $\frac{3}{100}$, which is greater than zero, and the two diagonal elements of the Hessian matrix are less than zero. As a result, the critical values $x_1 = 32$ and $x_2 = 25$ indicates a maximum.

e. What is the optimal level of output?

When $x_1 = 32$ and $x_2 = 25$, the level of output is given by

$$\begin{aligned} f(32, 25) &= 32^{1/5} 25^{1/2} \\ &= 10 \end{aligned}$$

f. How much does the firm spend on inputs?

When $x_1 = 32$ and $x_2 = 25$, the cost of the inputs is given by

$$\begin{aligned} w_1 x_1 + w_2 x_2 &= 5 \times 32 + 16 \times 25 \\ &= 560 \end{aligned}$$

g. Show that the marginal value product of x_1 at its optimal value is equal to w_1?

The marginal value product of x_1 is given by

$$\begin{aligned} p\frac{\partial f(x_1, x_2)}{\partial x_1} &= p\frac{1}{5}x_1^{-4/5}x_2^{1/2} = 80\frac{1}{5}x_1^{-4/5}x_2^{1/2} \\ &= 16x_1^{-4/5}x_2^{1/2} \end{aligned} \tag{17}$$

Then when $x_1 = 32, x_2 = 25$, the marginal value product of x_1 is given by

$$\begin{aligned} 16x_1^{-4/5}x_2^{1/2} &= 16 \times 32^{-4/5} 25^{1/2} = 16 \times \frac{1}{16} \times 5 \\ &= 5 \end{aligned}$$

h. Explain in words why the value of the marginal product for each input for this firm is equal to the price of that input at the profit maximizing level of input use for that input.

Setting the profit with respect to each input to zero in order to maximize profit is equivalent to set the value of the marginal product for each input to equal to the price of that input.

In other words, regarding one input level, given other inputs constant, if the value of marginal product is higher than the price of that input, the firm can profit more by increasing this input level. On the other hand, if the value of marginal product is less than the price of that input, the firm need to profit more from decreasing this input level. As a result, the firm attain its profit maximum when the value of the marginal product for each input is equal to the price of that input.

Problem 4. Consider a firm with a production function given by

$$f(x_1, x_2) = x_1^{1/5} x_2^{1/2}$$

The firm faces prices and a cost constraint given by

$$w_1 = 5$$

$$w_2 = 16$$

$$c_0 = 560$$

Find potential levels of x_1, x_2 and λ to maximize output for this firm given the cost constraint and the stated prices. Verify that these input levels maximize output.

a. Set up the objective function for this problem and find all first and second partial derivatives of the function with respect to x_1 and x_2.

$$\mathcal{L}(x_1, x_2, \lambda) \ = \ x_1^{1/5} x_2^{1/2} - \lambda(5x_1 + 16x_2 - 560) \tag{18}$$

$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = \frac{1}{5} x_1^{-4/5} x_2^{1/2} - 5\lambda$	$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = \frac{1}{2} x_1^{1/5} x_2^{-1/2} - 16\lambda$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -\frac{4}{25} x_1^{-9/5} x_2^{1/2}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = \frac{1}{10} x_1^{-4/5} x_2^{-1/2}$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = \frac{1}{10} x_1^{-4/5} x_2^{-1/2}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = -\frac{1}{4} x_1^{1/5} x_2^{-3/2}$

b. What is the derivative of the objective function in this problem with respect to λ?

$$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = -(5x_1 + 16x_2 - 560)$$

c. Find the partial derivatives of the constraint equation with respect to x_1 and x_2.

$\frac{\partial g(x_1,x_2)}{\partial x_1} = 5$	$\frac{\partial g(x_1,x_2)}{\partial x_2} = 16$

d. Use the information from 4a and 4b to find critical values for x_1, x_2 and λ.

By setting the first derivative of equation (18) to zero, we obtain

$$\frac{1}{5}x_1^{-4/5}x_2^{1/2} - 5\lambda = 0 \tag{19}$$
$$\frac{1}{2}x_1^{1/5}x_2^{-1/2} - 16\lambda = 0 \tag{20}$$
$$-(5x_1 + 16x_2 - 560) = 0 \tag{21}$$

Rearrange equation (19) and (20).

$$\frac{1}{5}x_1^{-4/5}x_2^{1/2} = 5\lambda \tag{22}$$
$$\frac{1}{2}x_1^{1/5}x_2^{-1/2} = 16\lambda \tag{23}$$

Divide the left side of equation (22) by the left side of equation (23) and divide the right side of equation (22) by the right side of equation (23).

$$\frac{\frac{1}{5}x_1^{-4/5}x_2^{1/2}}{\frac{1}{2}x_1^{1/5}x_2^{-1/2}} = \frac{5\lambda}{16\lambda}$$
$$\Rightarrow \quad x_1 = \frac{32}{25}x_2$$

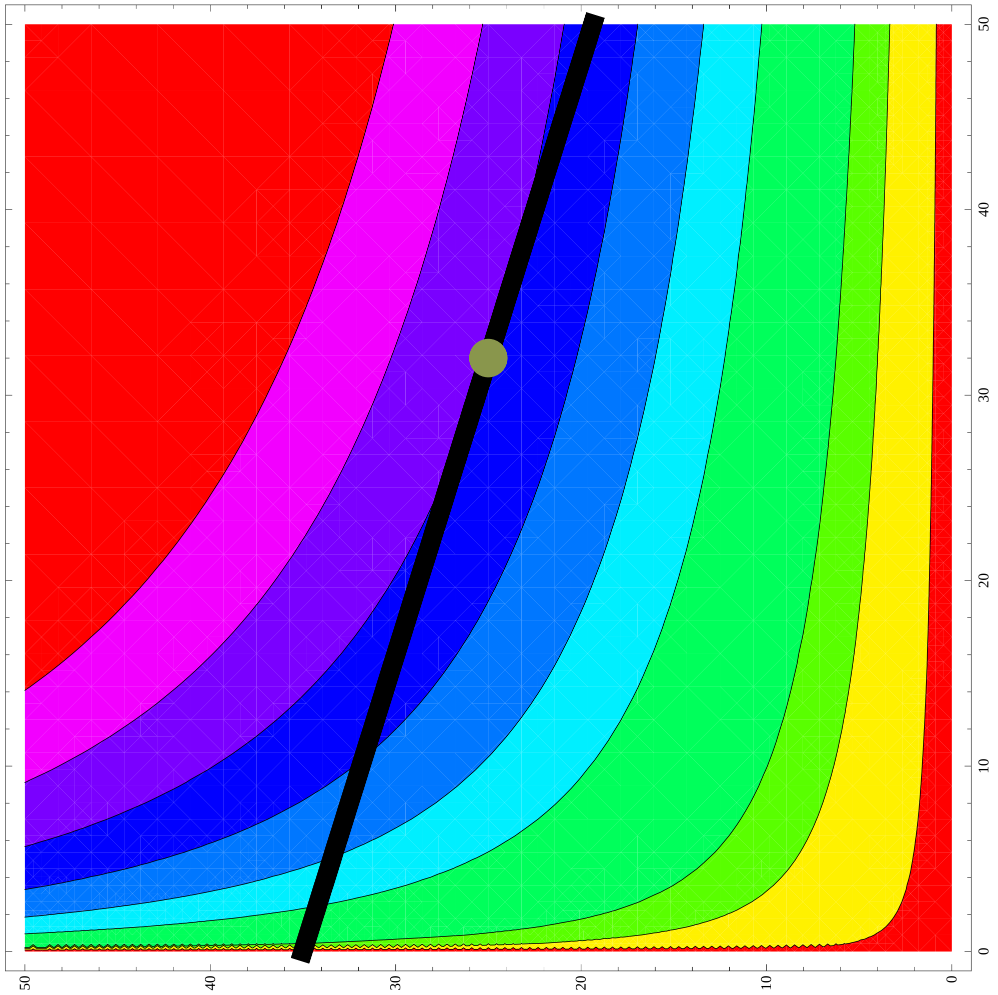
Substitute $x_1 = \dfrac{32}{25}x_2$ into equation (21).

$$\begin{aligned}
-(5x_1 + 16x_2 - 560) &= 0 \\
\Rightarrow \quad 5 \times \tfrac{32}{25}x_2 + 16x_2 - 560 &= 0 \\
\Rightarrow \quad \tfrac{32}{5}x_2 + 16x_2 - 560 &= 0 \\
\Rightarrow \quad \tfrac{4}{5}x_2 + 2x_2 - 70 &= 0 \\
\Rightarrow \quad \tfrac{14}{5}x_2 &= 70 \\
\Rightarrow \quad x_2 &= 25
\end{aligned}$$

Then $x_1 = \frac{32}{25}x_2 = 32$.
Substitute $x_1 = 32$, $x_2 = 25$ into equation (19).

$$\begin{aligned}
\tfrac{1}{5}x_1^{4/5}x_2^{1/2} - 5\lambda &= 0 \\
\Rightarrow \quad \tfrac{1}{5} \times 32^{-4/5} \times 25^{1/2} - 5\lambda &= 0 \\
\Rightarrow \quad \tfrac{1}{5} \times \tfrac{1}{16} \times 5 - 5\lambda &= 0 \\
\Rightarrow \quad \lambda &= \tfrac{1}{80}
\end{aligned}$$

Figure 5. Output Maximization Subject to a Cost Constraint



e. Use the answers from part 4d and the expressions from parts 4a and 4c to fill in the bordered Hessian matrix for this problem. Then determine whether the critical values indicate a maximum or a minimum. The determinant of the bordered Hessian is $\frac{7}{10}$.

Hints: $\frac{256}{640} = \frac{3}{10}$

$$H_B = \begin{vmatrix} \begin{aligned} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} &= -\tfrac{4}{25} x_1^{-9/5} x_2^{1/2} \\ &= -\tfrac{4}{25} \times 32^{-9/5} \times 25^{1/2} \\ &= -\tfrac{1}{640} \end{aligned} & \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = \frac{1}{800} & \frac{\partial g(x_1, x_2)}{\partial x_1} = 5 \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = \frac{1}{800} & \begin{aligned} \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} &= -\tfrac{1}{4} x_1^{1/5} x_2^{-3/2} \\ &= -\tfrac{1}{4} \times 32^{1/5} \times 25^{-3/2} \\ &= -\tfrac{1}{4} \times 32^{1/5} \times 25^{-3/2} \\ &= -\tfrac{1}{250} \end{aligned} & \frac{\partial g(x_1, x_2)}{\partial x_2} = 16 \\ \frac{\partial g(x_1, x_2)}{\partial x_1} = 5 & \frac{\partial g(x_1, x_2)}{\partial x_2} = 16 & 0 \end{vmatrix} = \begin{vmatrix} -\frac{1}{640} & \frac{1}{800} & 5 \\ \frac{1}{800} & -\frac{1}{250} & 16 \\ 5 & 16 & 0 \end{vmatrix}$$

$$= 5 \times \begin{vmatrix} \frac{1}{800} & 5 \\ -\frac{1}{250} & 16 \end{vmatrix} - 16 \times \begin{vmatrix} -\frac{1}{640} & 5 \\ \frac{1}{800} & 16 \end{vmatrix} = 5 \times \left(\frac{1}{50} + \frac{1}{50} \right) + 16 \times \left(\frac{1}{40} + \frac{1}{160} \right) = \frac{1}{5} + \frac{1}{2}$$

$$= \frac{7}{10} > 0$$

So this set of $x_1 = 32$, $x_2 = 25$, and $\lambda = \frac{1}{80}$ indicates a maximum.

f. How much output can this firm produce given it spends only \$560?

Given the conation of spending only \$560, to maximum output, $x_1 = 32$ and $x_2 = 25$. And hence the output is given by

$$f(32, 25) = 32^{1/5} \times 25^{1/2} = 2 \times 5 = 10$$

g. What is the marginal product of x_1 at its optimal value?
The marginal product of x_1 at the optimal value is given by

$$\left.\frac{\partial f(x_1, x_2)}{\partial x_1}\right|_{x_1=32, x_2=25} = \left.\frac{1}{5} x_1^{-4/5} x_2^{1/2}\right|_{x_1=32, x_2=25} = \frac{1}{5} \times 32^{-4/5} \times 25^{1/2}$$
$$= \frac{1}{16}$$

h. What is the marginal product of x_2 at its optimal value?

The marginal product of x_2 at the optimal value is given by

$$\left.\frac{\partial f(x_1, x_2)}{\partial x_2}\right|_{x_1=32, x_2=25} = \left.\frac{1}{2} x_1^{1/5} x_2^{-1/2}\right|_{x_1=32, x_2=25} = \frac{1}{2} \times 32^{1/5} \times 25^{-1/2}$$
$$= \frac{1}{5}$$

i. Show that the ratio of the marginal products is equal to the input price ratio?

The ratio of the marginal products is given by

$$\textit{Ratio of the marginal products} = \frac{1/16}{1/5} = \frac{5}{16} = \frac{w_1}{w_2}$$

So the ratio of the marginal products is equal to the input price ratio at the optimal level.

j. Interpret the condition in part i.

Equation (18) is case of more general form given below.

$$\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda(w_1 x_1 + w_2 x_2 - c_0) \tag{24}$$

In order to maximize $f(x_1, x_2)$ given the constraint of cost, we set the first derivative of equation (24) to zero and then we obtain

$$\frac{\partial f(x_1,x_2)}{\partial x_1} - \lambda w_1 = 0 \tag{25}$$

$$\frac{\partial f(x_1,x_2)}{\partial x_2} - \lambda w_2 = 0 \tag{26}$$

$$w_1 x_1 + w_2 x_2 - c_0 = 0 \tag{27}$$

Rearrange equation (25) and (26), we obtain

$$\frac{\partial f(x_1,x_2)}{\partial x_1} \Big/ w_1 = \lambda \tag{28}$$

$$\frac{\partial f(x_1,x_2)}{\partial x_2} \Big/ w_2 = \lambda \tag{29}$$

And from above two equations, we reach below condition.

$$\begin{aligned} \frac{\partial f(x_1,x_2)}{\partial x_1} \Big/ w_1 &= \frac{\partial f(x_1,x_2)}{\partial x_2} \Big/ w_2 \\ \Rightarrow \quad \frac{\partial f(x_1,x_2)}{\partial x_1} \Big/ \frac{\partial f(x_1,x_2)}{\partial x_2} &= w_1 \Big/ w_2 \end{aligned} \tag{30}$$

So at the optimal level of maximizing production, the ratio of the marginal products is equal to the ratio of input prices.

In other words, given the cost constraint, in order to maximize the production, consider to choose which inputs should be preferred to. Regarding input 1, the marginal product divided by its price means what would be produced if we choose to spend $1 on input 1. Similarly, regarding input 2, the marginal product divided by its price means what would be produced if we choose to spend $1 on input 2. So if the former quotient is higher than the latter, we prefer to choose input 1, but if the former is less than the latter, we prefer to choose input 2. Consequently, when they are equal, we reach an optimal level of maximizing production. And that the two quotients equal to each other is equivalent to that the ratio of the marginal products is equal to the input price ratio.

Problem 5. Consider a producer with a production function given by

$$f(x_1, x_2) \;=\; x_1^{1/5} x_2^{1/2}$$

The firm faces prices and an output target given by

$$w_1 \;=\; 5, \qquad w_2 \;=\; 16, \qquad y_0 \;=\; 10$$

Find potential levels of x_1 and x_2 to minimize the cost for this producer to reach the target level of output given the state prices. Verify that these input levels minimize cost.

a. Set up the objective function for this problem and find all first and second partial derivatives of the function with respect to x_1 and x_2.

$$\mathcal{L}(x_1, x_2, \lambda) = 5x_1 + 16x_2 - \lambda\left(x_1^{1/5} x_2^{1/2} - 10\right) \tag{31}$$

$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1} = 5 - \frac{1}{5}\lambda x_1^{-4/5} x_2^{1/2}$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2} = 16 - \frac{1}{2}\lambda x_1^{1/5} x_2^{-1/2}$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = -\frac{4}{25}\lambda x_1^{-9/5} x_2^{1/2}$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} = -\frac{1}{10}\lambda x_1^{-4/5} x_2^{-1/2}$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = -\frac{1}{10}\lambda x_1^{-4/5} x_2^{-1/2}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} = \frac{1}{4}\lambda x_1^{1/5} x_2^{-3/2}$

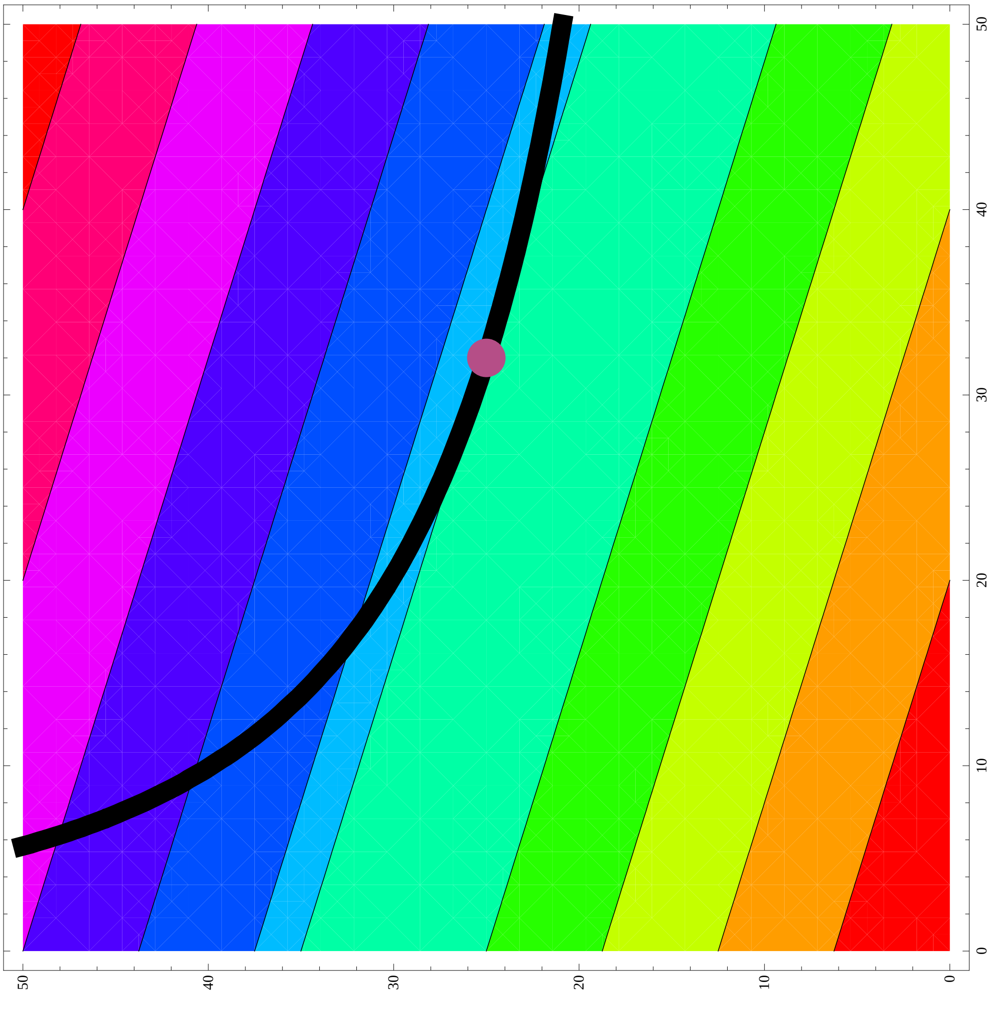
b. What is the derivative of the objective function in this problem with respect to λ?

$$\frac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda} = -\left(x_1^{1/5} x_2^{1/2} - 10\right)$$

c. Find the partial derivatives of the constraint equation with respect to x_1 and x_2.

$\frac{\partial g(x_1, x_2)}{\partial x_1} = \frac{1}{5} x_1^{-4/5} x_2^{1/2}$	$\frac{\partial g(x_1, x_2)}{\partial x_2} = \frac{1}{2} x_1^{1/5} x_2^{-1/2}$

Figure 6. Cost Minimization Subject to an Output Constraint



d. Use the information from 5a and 5b to find critical values for x_1, x_2 and λ.

By setting the first derivative of equation (31) to zero, we obtain

$$5 - \frac{1}{5}\lambda x_1^{-4/5} x_2^{1/2} = 0 \tag{32}$$

$$16 - \frac{1}{2}\lambda x_1^{1/5} x_2^{-1/2} = 0 \tag{33}$$

$$-\left(x_1^{1/5} x_2^{1/2} - 10\right) = 0 \tag{34}$$

Rearrange equation (32), (33), and (34).

$$\frac{1}{5}\lambda x_1^{-4/5} x_2^{1/2} = 5 \tag{35}$$

$$\frac{1}{2}\lambda x_1^{1/5} x_2^{-1/2} = 16 \tag{36}$$

$$x_1^{1/5} x_2^{1/2} - 10 = 0 \tag{37}$$

Divide the two sides of equation (35) by the two sides of equation (36) respectively.

$$\begin{aligned} \frac{\frac{1}{5}\lambda x_1^{-4/5} x_2^{1/2}}{\frac{1}{2}\lambda x_1^{1/5} x_2^{-1/2}} &= \frac{5}{16} \\ \Rightarrow \quad \frac{2x_2}{5x_1} &= \frac{5}{16} \\ \Rightarrow \quad x_1 &= \frac{32}{25}x_2 \end{aligned}$$

Substitute $x_1 = \frac{32}{25}x_2$ into equation (37).

$$\begin{aligned} x_1^{1/5} x_2^{1/2} - 10 &= 0 \\ \Rightarrow \quad \left(\tfrac{32}{25}\right)^{1/5} x_2^{1/5+1/2} &= 10 \\ \Rightarrow \quad 2\left(\tfrac{1}{25}\right)^{1/5} x_2^{7/10} &= 10 \\ \Rightarrow \quad 5^{-2/5} x_2^{7/10} &= 5 \\ \Rightarrow \quad x_2^{7/10} &= 5^{1+2/5} = 5^{7/5} \\ \Rightarrow \quad x_2 &= \left(5^{7/5}\right)^{10/7} = 25 \end{aligned}$$

Then $x_1 = \frac{32}{25}x_2 = 32$.
Substitute $x_1 = 32$, $x_2 = 25$ into equation (32).

$$
\begin{aligned}
5 - \tfrac{1}{5}\lambda x_1^{-4/5} x_2^{1/2} &= 0 \\
\Rightarrow \quad \tfrac{1}{5}\lambda 32^{-4/5} \times 25^{1/2} &= 5 \\
\Rightarrow \quad \lambda &= 5 \times 32^{4/5} = 80
\end{aligned}
$$

e. Substitute the values for x_1, x_2 and λ into the bordered Hessian matrix. Show that the determinant of this matrix is $-\frac{7}{800}$.

Hints: $16 \times 25 \times 16 = 6400$; $\lambda = 80$.

$$H_B = \begin{vmatrix} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} = \frac{1}{8} & \begin{aligned} \frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} &= -\tfrac{1}{10}\lambda x_1^{-4/5} x_2^{-1/2} \\ &= -\tfrac{1}{10} \times 80 \times 32^{-4/5} \times 25^{-1/2} \\ &= -\tfrac{1}{10} \end{aligned} & \frac{\partial g(x_1,x_2)}{\partial x_1} = \frac{1}{16} \\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} = -\frac{1}{10} & \begin{aligned} \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} &= \tfrac{1}{4}\lambda x_1^{1/5} x_2^{-3/2} \\ &= \tfrac{1}{4} \times 80 \times 32^{1/5} \times 25^{-3/2} \\ &= \tfrac{8}{25} \end{aligned} & \begin{aligned} \frac{\partial g(x_1,x_2)}{\partial x_2} &= \tfrac{1}{2} x_1^{1/5} x_2^{-1/2} \\ &= \tfrac{1}{2} \times 32^{1/5} \times 25^{-1/2} \\ &= \tfrac{1}{5} \end{aligned} \\ \begin{aligned} \frac{\partial g(x_1,x_2)}{\partial x_1} &= \tfrac{1}{5} x_1^{-4/5} x_2^{1/2} \\ &= \tfrac{1}{5} \times 32^{-4/5} \times 25^{1/2} \\ &= \tfrac{1}{16} \end{aligned} & \frac{\partial g(x_1,x_2)}{\partial x_2} = \frac{1}{5} & 0 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{8} & -\frac{1}{10} & \frac{1}{16} \\ -\frac{1}{10} & \frac{8}{25} & \frac{1}{5} \\ \frac{1}{16} & \frac{1}{5} & 0 \end{vmatrix} = \frac{1}{16} \times \begin{vmatrix} -\frac{1}{10} & \frac{1}{16} \\ \frac{8}{25} & \frac{1}{5} \end{vmatrix} - \frac{1}{5} \times \begin{vmatrix} \frac{1}{8} & \frac{1}{16} \\ -\frac{1}{10} & \frac{1}{5} \end{vmatrix} = -\frac{1}{16} \times \frac{1}{25} - \frac{1}{5} \times \frac{5}{160}$$

$$= -\frac{7}{800}$$

f. How much does this firm spend on inputs?

When $x_1 = 32$ and $x_2 = 25$, the cost of inputs is given by

$$5x_1 + 16x_2 = 5 \times 32 + 16 \times 25 = 560$$

g. How much output does it produce?

When $x_1 = 32$ and $x_2 = 25$, the output is given by

$$\begin{aligned} f(32, 25) &= 32^{1/5} \times 25^{1/2} = 2 \times 5 \\ &= 10 \end{aligned}$$

h. What is the marginal product of x_1 at its optimal value?

$$\begin{aligned} \left.\frac{\partial f(x_1, x_2)}{\partial x_1}\right|_{x_1=32, x_2=25} &= \left.\frac{1}{5} x_1^{-4/5} x_2^{1/2}\right|_{x_1=32, x_2=25} = \frac{1}{5} \times 32^{-4/5} \times 25^{1/2} \\ &= \frac{1}{16} \end{aligned}$$

i. What is the marginal product of x_2 at its optimal value?

$$\begin{aligned} \left.\frac{\partial f(x_1, x_2)}{\partial x_1}\right|_{x_1=32, x_2=25} &= \left.\frac{1}{5} x_1^{-4/5} x_2^{1/2}\right|_{x_1=32, x_2=25} = \frac{1}{5} \times 32^{-4/5} \times 25^{1/2} \\ &= \frac{1}{16} \end{aligned}$$

j. Show that the ratio of the marginal products is equal to the input price ratio?

The ratio of the marginal products is given by

$$\textit{Ratio of the marginal products} = \frac{1/16}{1/5} = \frac{5}{16} = \frac{w_1}{w_2}$$

So the ratio of the marginal products is equal to the input price ratio at the optimal level.

k. Interpret the condition in part j.

Equation (31) is a special case of more general from given below.

$$\mathcal{L}(x_1, x_2, \lambda) = w_1 x_1 + w_2 x_2 - \lambda\left(f(x_1, x_2) - y_0\right) \tag{38}$$

Given the constraint of output, setting the first derivative of equation (38) to minimize the cost, we obtain

$$w_1 - \lambda \frac{\partial f(x_1, x_2)}{\partial x_1} = 0 \tag{39}$$

$$w_2 - \lambda \frac{\partial f(x_1, x_2)}{\partial x_2} = 0 \tag{40}$$

$$f(x_1, x_2) - y_0 = 0 \tag{41}$$

Rearrange equation (39) and (40), we obtain

$$\frac{1}{\lambda} = \frac{\partial f(x_1, x_2)}{\partial x_1} \Big/ w_1 \tag{42}$$

$$\frac{1}{\lambda} = \frac{\partial f(x_1, x_2)}{\partial x_2} \Big/ w_1 \tag{43}$$

And from above two equations, we reach below condition.

$$\begin{aligned} \frac{\partial f(x_1, x_2)}{\partial x_1} \Big/ w_1 &= \frac{\partial f(x_1, x_2)}{\partial x_2} \Big/ w_2 \\ \Rightarrow \frac{\partial f(x_1, x_2)}{\partial x_1} \Big/ \frac{\partial f(x_1, x_2)}{\partial x_2} &= w_1 \Big/ w_2 \end{aligned} \tag{44}$$

So at the optimal level of minimizing cost, the ratio of the marginal products is equal to the ratio of input prices.

In other words, given the constant output, in order to minimize the cost, consider the cost of one input given the other unchanged in order to produce one unit of output, which also is the input price divided by marginal product. So if the input price divided by marginal product of input 1 is higher than the input price divided by marginal product of input 2, cost can be decreased by switching to more input 2. On the other hand, if the input price divided by marginal product of input 1 is less than the input price divided by marginal product of input 2, cost can be decreased by switching to more input 1. As a result, when the two quotients are equal to each other, the cost attains its minimization. And this condition is equivalent to that the ratio of the marginal products is equal to the input price ratio.

l. Why did you not need to do any of the work in problem 5 given you had already worked problems 3 and 4?

When $x_1 = 32$ and $x_2 = 25$, we not only know that the production is 10 but also the ratio of marginal products is equal to the ratio of input prices. And these are just the conditions that we need to minimize the cost given the output constraint for this problem.