ECONOMICS 207
SPRING 2008
LABORATORY EXERCISE 5
KEY

Problem 1. Solve the following systems of equations for x_1 and x_2 using the method of substitution.
a.

$$\{x_1 = 64,\ x_2 = 32\}$$

$$20x_1^{-2/3}x_2^{2/5} - 5 = 0$$

$$24x_1^{1/3}x_2^{-3/5} - 12 = 0$$

From the second equation,

$$\begin{aligned} 24x_1^{1/3}x_2^{-3/5} - 12 &= 0 \\ \Rightarrow \quad x_1^{1/3}x_2^{-3/5} &= 12/24 = \tfrac{1}{2} \\ \Rightarrow \quad x_1^{1/3} &= \tfrac{1}{2}x_2^{3/5} \\ \Rightarrow \quad x_1^{-2/3} &= 4x_2^{-6/5} \end{aligned}$$

Subsitute $x_1^{-2/3} = 4x_2^{-6/5}$ into the first equation.

$$\begin{aligned} 20x_1^{-2/3}x_2^{2/5} - 5 &= 0 \\ \Rightarrow \quad 20 \cdot 4x_2^{-6/5}x_2^{2/5} &= 5 \\ \Rightarrow \quad x_2^{-4/5} &= 5/80 = 1/16 \\ \Rightarrow \quad x_2^{4/5} &= 16 \\ \Rightarrow \quad x_2 &= 16^{5/4} \\ \Rightarrow \quad x_2 &= 32 \end{aligned}$$

Subsitute $x_2 = 32$ into $x_1^{-2/3} = 4x_2^{-6/5}$.

$$\begin{aligned} x_1^{-2/3} &= 4x_2^{-6/5} \\ \Rightarrow \quad x_1^{-2/3} &= 4 \times 32^{-6/5} \\ \Rightarrow \quad x_1^{2/3} &= \tfrac{1}{4} \times 32^{6/5} \\ \Rightarrow \quad x_1^{2/3} &= 16 \\ \Rightarrow \quad x_1 &= 64 \end{aligned}$$

So the solution is

$$x_1 = 64,\ x_2 = 32.$$

Date: July 2, 2008.

b.

$$\{x_1 = 49,\ x_2 = 32\}$$

$$\begin{aligned} 70x_1^{-1/2}x_2^{2/5} - 40 &= 0 \\ 56x_1^{1/2}x_2^{-3/5} - 49 &= 0 \end{aligned}$$

From the second equation,

$$\begin{aligned} 56x_1^{1/2}x_2^{-3/5} - 49 &= 0 \\ \Rightarrow \quad x_1^{1/2}x_2^{-3/5} &= 49/56 = 7/8 \\ \Rightarrow \quad x_1^{1/2} &= \tfrac{7}{8}x_2^{3/5} \\ \Rightarrow \quad x_1^{-1/2} &= \tfrac{8}{7}x_2^{-3/5} \end{aligned}$$

Substitute $x_1^{-1/2} = \frac{8}{7}x_2^{-3/5}$ into the first equation.

$$\begin{aligned} 70x_1^{-1/2}x_2^{2/5} - 40 &= 0 \\ \Rightarrow \quad 70 \cdot \tfrac{8}{7}x_2^{-3/5}x_2^{2/5} - 40 &= 0 \\ \Rightarrow \quad 80x_2^{-1/5} &= 40 \\ \Rightarrow \quad x_2^{1/5} &= 2 \\ \Rightarrow \quad x_2 &= 32 \end{aligned}$$

Substitute $x_2 = 32$ into $x_1^{-1/2} = \frac{8}{7}x_2^{-3/5}$.

$$\begin{aligned} x_1^{-1/2} &= \tfrac{8}{7}x_2^{-3/5} \\ \Rightarrow \quad x_1^{-1/2} &= \tfrac{8}{7} \cdot 32^{-3/5} = \tfrac{1}{7} \\ \Rightarrow \quad x_1 &= 49 \end{aligned}$$

So the solution is

$$x_1 = 49,\ x_2 = 32.$$

Problem 2. Solve the following systems of equations for x_1 and x_2 first using the method of substitution and then using the method of elimination.

a.

$$\{x_1 = -1, \ x_2 = -5\}$$

$$\begin{aligned} x_1 - 3x_2 &= 14 \\ 3x_1 - 10x_2 &= 47 \end{aligned}$$

Methods of substitution

From the first equation,

$$x_1 - 3x_2 = 14 \ \Rightarrow \ x_1 = 3x_2 + 14.$$

Substitute $x_1 = 3x_2 + 14$ into the second equation.

$$\begin{aligned} 3x_1 - 10x_2 &= 47 \\ \Rightarrow \quad 3(3x_2 + 14) - 10x_2 &= 47 \\ \Rightarrow \quad 9x_2 + 42 - 10x_2 &= 47 \\ \Rightarrow \quad x_2 &= -5 \end{aligned}$$

Substitute $x_2 = -5$ into $x_1 = 3x_2 + 14$.

$$\begin{aligned} x_1 &= 3x_2 + 14 \\ &= -3 \times 5 + 14 = -1 \end{aligned}$$

So the solution is $x_1 = -1, \ x_2 = -5$.

Methods of elimation

Multiply the first equation by -3 and add it to the second equation.

$$\begin{aligned} -3x_1 + 9x_2 + 3x_1 - 10x_2 &= -3 \times 14 + 47 \\ \Rightarrow \quad -x_2 &= 5 \\ \Rightarrow \quad x_2 &= -5 \end{aligned}$$

Multiply the first equation by -10 and add it to the second equation multiplied by 3.

$$\begin{aligned} -10x_1 + 30x_2 + 9x_1 - 30x_2 &= -14 \times 10 + 47 \times 3 \\ \Rightarrow \quad -x_1 &= 1 \\ \Rightarrow \quad x_1 &= -1 \end{aligned}$$

So the solution is $x_1 = -1, \ x_2 = -5$.

b.

$$\{x_1 \;=\; 10,\; x_2 \;=\; 3\}$$

$$\begin{aligned} 3x_1 \;+\; 16x_2 \;&= 78 \\ 4x_1 \;+\; 21x_2 \;&= 103 \end{aligned}$$

Methods of substitution

From the first equation,

$$3x_1 + 16x_2 = 78 \;\Rightarrow\; x_1 = (78 - 16x_2)/3.$$

Substitute $x_1 = (78 - 16x_2)/3$ into the second equation.

$$\begin{aligned} 4x_1 + 21x_2 &= 103 \\ \Rightarrow \quad 4(78 - 16x_2)/3 + 21x_2 &= 103 \\ \Rightarrow \quad 4(78 - 16x_2) + 63x_2 &= 309 \\ \Rightarrow \quad 312 - 64x_2 + 63x_2 &= 309 \\ \Rightarrow \quad x_2 &= 3 \end{aligned}$$

Substitute $x_2 = 3$ into $x_1 = (78 - 16x_2)/3$.

$$\begin{aligned} x_1 &= (78 - 16x_2)/3 \\ &= (78 - 48)/3 = 10 \end{aligned}$$

So the solution is $x_1 = 10,\; x_2 = 3$.

Methods of elimation

Multiply the first equation by -4 and add it to the second equation multiplied by 3.

$$\begin{aligned} -12x_1 - 64x_2 + 12x_1 + 63x_2 &= 78 \times (-4) + 103 \times 3 \\ \Rightarrow \quad -x_2 &= -3 \\ \Rightarrow \quad x_2 &= 3 \end{aligned}$$

Multiply the first equation by -21 and add it to the second equation multiplied by 16.

$$\begin{aligned} -63x_1 + (16 \times 21)x_2 + 64x_1 - (21 \times 16)x_2 &= -78 \times 21 + 103 \times 16 \\ \Rightarrow \quad x_1 &= -1638 + 1648 \\ \Rightarrow \quad x_1 &= 10 \end{aligned}$$

So the solution is $x_1 = 10,\; x_2 = 3$.

Problem 3. Solve the following system of equations for x_1, x_2, and x_3 first using the method of substitution and then using the method of elimination.

$$\{x_1 = 2,\ x_2 = 2,\ x_3 = -1\}$$

$$-x_1 + x_2 - 3x_3 = 3$$

$$4x_1 - 5x_2 + 12x_3 = -14$$

$$6x_1 - 2x_2 + 19x_3 = -11$$

Method of substitution

From the first equation,

$$-x_1 + x_2 - 3x_3 = 3 \Rightarrow x_1 = x_2 - 3x_3 - 3.$$

Substitute $x_1 = x_2 - 3x_3 - 3$ into the second and third equation.

$$\begin{cases} 4x_1 - 5x_2 + 12x_3 & = & -14 \\ 6x_1 - 2x_2 + 19x_3 & = & -11 \end{cases}$$

$$\Rightarrow \begin{cases} 4(x_2 - 3x_3 - 3) - 5x_2 + 12x_3 & = & -14 \\ 6(x_2 - 3x_3 - 3) - 2x_2 + 19x_3 & = & -11 \end{cases}$$

$$\Rightarrow \begin{cases} 12 - x_2 & = & -14 \\ 4x_2 + x_3 & = & 7 \end{cases}$$

$$\Rightarrow \begin{cases} x_2 & = & 2 \\ x_3 & = & -1 \end{cases}$$

Then,

$$x_1 = x_2 - 3x_3 - 3 = 2 + 3 - 3 = 2.$$

So the solution is

$$x_1 = 2,\ x_2 = 2,\ x_3 = -1.$$

Method of Elimination

Multiply the first equation by 4 and add it to the second equation.

$$-4x_1 + 4x_2 - 12x_3 + 4x_1 - 5x_2 + 12x_3 = 12 - 14$$

That is,

$$-x_2 = -2 \ \Rightarrow \ x_2 = 2.$$

Multiply the first equation by 6 and add it to the third equation.

$$-6x_1 + 6x_2 - 18x_3 + 6x_1 - 2x_2 + 19x_3 = 18 - 11$$

That is,

$$4x_2 + x_3 = 7 \tag{1}$$

Multiply $x_2 = 2$ by -4 and add it to (1).

$$\begin{aligned} -4x_2 + 4x_2 + x_3 &= -8 + 7 \\ \Rightarrow \qquad x_3 &= -1 \end{aligned}$$

Then, add $x_2 = 2$ multiplied by -1 and $x_3 = -1$ multiplied by 3 to the first equation.

$$\begin{aligned} -x_1 + x_2 - 3x_3 - x_2 + 3x_3 &= 3 - 2 + (-3) \\ \Rightarrow \qquad x_1 &= 2 \end{aligned}$$

So the solution is

$$x_1 = 2, \ x_2 = 2, \ x_3 = -1.$$

Problem 4. Find the derivatives of each of the following functions with respect to x.

a. $y = 3x^2 + 2x^4$

$$\frac{dy}{dx} = 6x + 8x^3.$$

b. $f(x) = 3x^2 + 5e^x$

$$f'(x) = 6x + 5e^x$$

c. $f(x) = 5x^3 - 2\log[x]$

$$f'(x) = 15x^2 - \frac{2}{x}$$

d. $f(x) = -3x^4 + 2x^3 + 3^x$

$$f'(x) = -12x^3 + 6x^2 + 3^x \ln 3$$

e. $f(x) = 6x^{1/2} + 9x^{1/3} - 3x^{-2}$

$$f'(x) = 3x^{-1/2} + 3x^{-2/3} + 6x^{-3}$$

f. $f(x) = 4x^{-2} - 3xe^x$

$$f'(x) = -8x^{-3} - 3e^x - 3xe^x$$

g. $f(x) = 12x^{1/2} + 2x^3 \log[x]$

$$\begin{aligned} f' &= 6x^{-1/2} + 6x^2 \log[x] + 2x^3 \frac{1}{x} \\ &= 6x^{-1/2} + 6x^2 \log[x] + 2x^2 \end{aligned}$$

h. $f(x) = (3x + 2)^3$ Find in two different ways.

Method 1

$$\begin{aligned} f'(x) &= 3(3x+2)^2 \cdot 3 \\ &= 9(3x+2)^2 \end{aligned}$$

Method 2

$$\begin{aligned} f(x) &= 27x^3 + 3 \times (9x^2) \times 2 + 3 \times (3x) \times 4 + 8 \\ &= 27x^3 + 54x^2 + 36x + 8. \end{aligned}$$

Then,

$$\begin{aligned} f'(x) &= 81x^2 + 108x + 36 \\ &= 9(9x^2 + 12x + 4) \\ &= 9(3x+2)^2 \end{aligned}$$

i. $f(x) = \frac{x^2+2x}{4x^2}$

$$\begin{aligned} f'(x) &= \frac{(2x+2) \cdot (4x^2) - (x^2+2x) \cdot (8x)}{16x^4} \\ &= \frac{8x^3 + 8x^2 - 8x^3 - 16x^2}{16x^4} \\ &= \frac{-8x^2}{16x^4} \\ &= \frac{-1}{2x^2} \end{aligned}$$

j. $f(x) = \frac{4x^2}{x^2+2x}$

$$f(x) = \frac{4x^2}{x^2+2x} = \frac{4x}{x+2} = 4 - \frac{8}{x+2}$$

$$\begin{aligned} f'(x) &= 8 \cdot (-\frac{1}{(x+2)^2}) \\ &= \frac{8}{(x+2)^2} \end{aligned}$$

Problem 5. Find the derivatives of each of the following functions with respect to x.

a. $f(x) = (x^2 + 4x)^2$

$$\begin{aligned} f'(x) &= 2(x^2 + 4x) \cdot (2x + 4) \\ &= 4x(x + 4)(x + 2) \end{aligned}$$

b. $f(x) = (5x - 3)(2x + 4)$ Show two ways.

Method 1

$$f(x) = (5x - 3)(2x + 4) = 10x^2 + 14x - 12$$

$$f'(x) = 20x + 14$$

Method 2

$$\begin{aligned} f'(x) &= 5(2x + 4) + (5x - 3) \cdot 2 = 10x + 20 + 10x - 6 \\ &= 20x + 14 \end{aligned}$$

c. $f(x) = 3e^{2x^2+3x}$

$$\begin{aligned} f'(x) &= 3e^{2x^2+3x} \cdot (4x + 3) \\ &= 9e^{2x^2+3x} + 12xe^{2x^2+3x} \end{aligned}$$

d. $f(x) = 3x^2 e^{2x^2+3x}$

$$\begin{aligned} f'(x) &= 6xe^{2x^2+3x} + 3x^2 e^{2x^2+3x} \cdot (4x + 3) \\ &= 6xe^{2x^2+3x} + 3x^2(4x + 3)e^{2x^2+3x} \end{aligned}$$

e. $f(x) \; = \; 12^x e^{2x^2+3x}$

$$
\begin{aligned}
f'(x) &= (12^x \ln 12) \cdot e^{2x^2+3x} + 12^x e^{2x^2+3x} \cdot (4x+3) \\
&= (\ln 12) 12^x e^{2x^2+3x} + 12^x (4x+3) e^{2x^2+3x}
\end{aligned}
$$

f. $f(x) \; = \; \log[(x^3 - 4x)^2]$

$$
\begin{aligned}
f'(x) &= \frac{1}{(x^3-4x)^2} \cdot [2(x^3-4x)(3x^2-4)] \\
&= \frac{2(3x^2-4)}{x^3-4x} \\
&= \frac{6x^2-8)}{x^3-4x}
\end{aligned}
$$

g. $f(x) \; = \; \frac{3xe^{2x}}{4x^2+2}$

$$
\begin{aligned}
f'(x) &= \frac{(3e^{2x} + 6xe^{2x})(4x^2+2) - 3xe^{2x}(8x)}{(4x^2+2)^2} \\
&= \frac{3e^{2x} + 6xe^{2x}}{4x^2+2} - \frac{24x^2 e^{2x}}{(4x^2+2)^2} \\
&= \frac{3e^{2x}}{4x^2+2} + \frac{6xe^{2x}}{4x^2+2} - \frac{24x^2 e^{2x}}{(4x^2+2)^2}
\end{aligned}
$$

h. $f(x) = \frac{3x\log[2x^2]}{x^2 + 2x}$

$$\begin{aligned} f(x) &= \frac{3x\log[2x^2]}{x^2+2x} \\ &= \frac{3\log[2x^2]}{x+2} \end{aligned}$$

$$\begin{aligned} f'(x) &= 3\,\frac{\frac{1}{2x^2}\cdot(4x)\cdot(x+2) - 3\log[2x^2]}{(x+2)^2} \\ &= \frac{6}{x(x+2)} - \frac{3\log[2x^2]}{(x+2)^2} \end{aligned}$$

i. $f(x) = 2x^4 + 2xe^{3x^2}$

$$\begin{aligned} f'(x) &= 8x^3 + 2e^{3x^2} + 2xe^{3x^2}\cdot(6x) \\ &= 8x^3 + 2e^{3x^2} + 12x^2e^{3x^2} \end{aligned}$$

j. $f(x) = \frac{3x^2e^x}{x^2+3}$

$$\begin{aligned} f'(x) &= \frac{(6xe^x + 3x^2e^x)(x^2+3) - 3x^2e^x\cdot(2x)}{(x^2+3)^2} \\ &= \frac{6xe^x + 3x^2e^x}{x^2+3} - \frac{6x^3e^x}{(x^2+3)^2} \end{aligned}$$

Problem 6. For each of the following, take the derivative with respect to x_1, set the derivative equal to zero and solve the resulting equation for x_1.

a. $\{x_1 \; = \; 256\}$

$f(x) \; = \; 256x_1^{3/8} \; - \; 3x_1$

$$f'(x) = 256 \times \frac{3}{8} x_1^{-5/8} - 3$$
$$= 96x_1^{-5/8} - 3$$

$$\begin{aligned} f'(x) &= 0 \\ \Rightarrow \quad 96x_1^{-5/8} - 3 &= 0 \\ \Rightarrow \quad x_1^{-5/8} &= 3/96 = 1/32 \\ \Rightarrow \quad x_1^{5/8} &= 32 \\ \Rightarrow \quad x_1 &= 32^{8/5} \\ \Rightarrow \quad x_1 &= 256 \end{aligned}$$

b. $\{x_1 \; = \; \frac{81}{16}\}$

$f(x) \; = \; 108x_1^{1/4} \; - \; 8x_1$

$$f'(x) = 108 \times \frac{1}{4} x_1^{-3/4} - 8$$
$$= 27x_1^{-3/4} - 8$$

$$\begin{aligned} f'(x) &= 0 \\ \Rightarrow \quad 27x_1^{-3/4} - 8 &= 0 \\ \Rightarrow \quad 27x_1^{-3/4} &= 8 \\ \Rightarrow \quad x_1^{3/4} &= 27/8 \\ \Rightarrow \quad x_1 &= (27/8)^{4/3} \\ \Rightarrow \quad x_1 &= \tfrac{81}{16} \end{aligned}$$

c. $f(x) \; = \; 75px_1^{1/3} \; - \; 4x_1$

$$
\begin{aligned}
f'(x) &= 75 \times \frac{1}{3} p x_1^{-2/3} - 4 \\
&= 25 p x_1^{-2/3} - 4
\end{aligned}
$$

$$
\begin{array}{rrcl}
 & f'(x) & = & 0 \\
\Rightarrow & 25px_1^{-2/3} - 4 & = & 0 \\
\Rightarrow & x_1^{-2/3} & = & 4/(25p) \\
\Rightarrow & x_1 & = & (4/(25p))^{-3/2} \\
\Rightarrow & x_1 & = & \frac{125}{8} p^{3/2}
\end{array}
$$