ECONOMICS 207
SPRING 2008
LABORATORY EXERCISE 6
KEY

Problem 1. Find the derivatives of each of the following functions with respect to x.

a. $y \;=\; 24x^{1/3} + 3x^2 e^{2x^3}$

$$\begin{aligned}\frac{dy}{dx} &= 24\frac{1}{3}x^{-2/3} + 6xe^{2x^3} + 3x^2(e^{2x^3} \cdot (6x^2)) \\ &= 8x^{-2/3} + 6xe^{2x^3} + 18x^4 e^{2x^3}\end{aligned}$$

b. $y \;=\; \log[(2x^3 - 4x^2)^4]$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{(2x^3 - 4x^2)^4} \cdot (4(2x^3 - 4x^2)^3) \cdot (6x^2 - 8x) \\ &= \frac{4(6x^2 - 8x)}{2x^3 - 4x^2} \\ &= \frac{12x - 16}{x^2 - 2x}\end{aligned}$$

Date: July 2, 2008.

c. $f(x) \;=\; \frac{2x^3(x^2+2x)^3}{4x^2-2x-7}$

$$
\begin{aligned}
f'(x) &= \frac{\left[6x^2(x^2+2x)^3 + 2x^3\cdot(3(x^2+2x)^2)\cdot(2x+2)\right](4x^2-2x-7) - [2x^3(x^2+2x)^3]\cdot(8x-2)}{(4x^2-2x-7)^2} \\
&= \frac{6x^2(x^2+2x)^3 + 2x^3\cdot(3(x^2+2x)^2)\cdot(2x+2)}{4x^2-2x-7} - \frac{[2x^3(x^2+2x)^3]\cdot(8x-2)}{(4x^2-2x-7)^2} \\
&= \frac{6x^2(x^2+2x)^3 + 6x^3\cdot(x^2+2x)^2\cdot(2x+2)}{4x^2-2x-7} - \frac{[2x^3(x^2+2x)^3]\cdot(8x-2)}{(4x^2-2x-7)^2}
\end{aligned}
$$

d. Find the derivative with respect to x_1. $\quad y \;=\; 1620x_1^{1/5}x_2^{2/5} \;-\; 16x_1 \;-\; 243x_2$

$$
\begin{aligned}
\frac{\partial y}{\partial x_1} &= 1620x_2^{2/5}\cdot\left(\frac{1}{5}x_1^{-4/5}\right) - 16 \\
&= 324x_1^{-4/5}x_2^{2/5} - 16
\end{aligned}
$$

Problem 2. Find the second derivative of each of the following functions with respect to x

a. $y \ = \ 5x^3 + 4x^2 - 10x$

$$\frac{dy}{dx} = 15x^2 + 8x - 10$$

$$\frac{d^2y}{dx^2} = 30x + 8$$

b. $y \ = \ (x^2 + 4x)^3$

$$\begin{aligned}\frac{dy}{dx} &= 3(x^2 + 4x)^2 \cdot (2x + 4)\\ &= 6(x^2 + 4x)^2(x + 2)\end{aligned}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= 6 \cdot [2(x^2 + 4x)(2x + 4)(x + 2) + (x^2 + 4x)^2]\\ &= 24(x^2 + 4x)(x + 2)^2 + 6(x^2 + 4x)^2\end{aligned}$$

c. $y = 5xe^{2x^3+4x}$

$$\frac{dy}{dx} = 5e^{2x^3+4x} + 5xe^{2x^3+4x} \cdot (6x^2+4)$$

$$\begin{aligned}
\frac{d^2y}{dx^2} &= 5e^{2x^3+4x}(6x^2+4) + \Big[5e^{2x^3+4x} + 5xe^{2x^3+4x} \cdot (6x^2+4)\Big] \cdot (6x^2+4) + 5xe^{2x^3+4x} \cdot (12x) \\
&= 5e^{2x^3+4x}\left[(12x^2+8) + x(6x^2+4)^2 + 12x^2\right] \\
&= 20e^{2x^3+4x}\left[6x^2+2+x(9x^4+12x^2+4)\right] \\
&= 20e^{2x^3+4x}\left(9x^5+12x^3+6x^2+4x+2\right)
\end{aligned}$$

d. $y = 20(100x + 20x^2 - x^3) - 500x$

$$\frac{dy}{dx} = 20(100+40x-3x^2) - 500$$

$$\begin{aligned}
\frac{d^2y}{dx^2} &= 20(40-6x) \\
&= -120x + 800
\end{aligned}$$

Problem 3. In the following problems you are given a production function for a firm where y is the level of output and x is the level of the variable input. You are given the price (p) of the output and the price (w) of the single variable input. For each problem

1) Write down an equation that represents profit for the firm.
2) Maximize this function by taking its derivative with respect to the variable input x and setting it equal to zero.
3) Solve for the profit maximizing level of x.
4) Find the optimal level of output.
5) What is revenue at the optimal level of output?
6) What is cost at the optimal level of the input x?

a.

$$output\ price \;=\; p \;= 20$$

$$input\ price \;=\; w \;= 500$$

$$y \;=\; output \;=\; f(x) \;=\; 100x \;+\; 20x^2 \;-x^3$$

For this example, profit is given by

$$\begin{aligned} Profit \;&= 20(100x \;+\; 20x^2 \;-x^3) \;-\; 500x \\ &= 2000x \;+\; 400x^2 \;-20x^3 \;-\; 500x \\ &= 1500x \;+\; 400x^2 \;-20x^3 \end{aligned}$$

For this example, the derivative of profit is given by

$$\begin{aligned} Profit \;&= 1500x \;+\; 400x^2 \;-20x^3 \\ \frac{d\,Profit}{dx} \;&= 1500 \;+\; 800x \;-\; 60x^2 \\ \frac{d^2\,Profit}{dx^2} \;&= 800 \;-\; 120x \end{aligned}$$

Now set the derivative of profit equal to zero and solve for x as follows.

$$\begin{aligned} \frac{d\,Profit}{dx} \;=\; 1500 \;+\; 800x \;-\; 60x^2 \;&= 0 \\ \Rightarrow\; 3x^2 - 40x - 75 \;&= 0 \\ \Rightarrow\; (3x+5)(x-15) \;&= 0 \\ \Rightarrow\; x \;=\; -\frac{5}{3} \text{ or } x = 15 \end{aligned}$$

At x = -$\frac{5}{3}$, $\frac{d^2\,Profit}{dx^2} \;=\; 800 \;-\; 120(-5/3) \;=\; 800 \;+\; 200 \;=\; 1000 > 0$.
At x = 15, $\frac{d^2\,Profit}{dx^2} \;=\; 800 \;-\; 120(15) \;=\; 800 \;-\; 1800 \;=\; -1000 < 0$. The optimal x =15.
The optimal output is given by

$$\begin{aligned} y \;&= 100x \;+\; 20x^2 \;-x^3 \\ &= (100)(15) + 20(15^2) - 15^3 \\ &= 1500 + (20)(225) - 3375 \\ &= 1500 + 4500 - 3375 \;=\; 6000 - 3375 = 2625 \end{aligned}$$

Revenue is equal to the optimal output multiplied by the output price.

$$revenue \;=\; py \;=\; (20)(2625) \;=\; 52500$$

Cost of production is equal to the level of the input used multiplied by the input price.

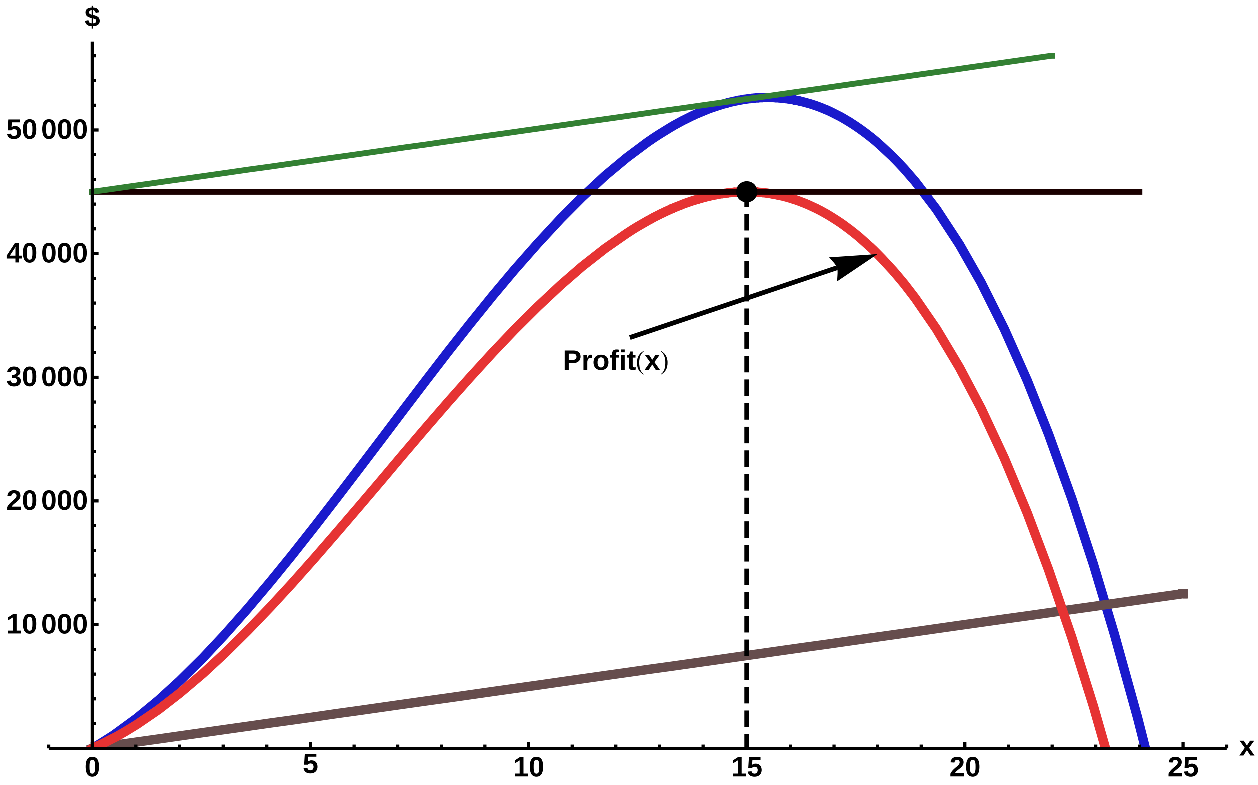
$$cost = wx = (500)(15) = 7500$$

Profit is equal to revenue minus cost.

$$profit = revenue - cost = 52500 - 7500 = 45000.$$

A graph of revenue, cost and profit is given in figure 1

Figure 1. Revenue, Cost and Profit



b.

$$\begin{aligned} output\ price &= p &= 2 \\ input\ price &= w &= 1998 \\ y = output &= f(x) &= 200x + 100x^2 - 3x^3 \end{aligned}$$

The profit is given by

$$\begin{aligned} Profit &= 2(200x + 100x^2 - 3x^3) - 1998x \\ &= -1598x + 200x^2 - 6x^3 \end{aligned}$$

The first and second derivative of the profit with respect to input x are given by

$$\begin{aligned} Profit &= -1598x + 200x^2 - 6x^3 \\ \frac{dProfit}{dx} &= -1598 + 400x - 18x^2 \qquad (1) \\ \frac{d^2Profit}{dx^2} &= 400 - 36x \end{aligned}$$

Now set the first derivative of the profit, equation (1), to zero and solve x as follows.

$$\begin{aligned} \frac{dProfit}{dx} &= -1598 + 400x - 18x^2 = 0 \\ \Rightarrow \quad & -1598 + 400x - 18x^2 = 0 \\ \Rightarrow \quad & -2(x-17)(9x-47) = 0 \\ \Rightarrow \quad & x = 17 \quad \text{or} \quad x = \frac{47}{9} \end{aligned}$$

At $x = \frac{47}{9}$, $\frac{d^2Profit}{dx^2} = 400 - 36\frac{47}{9} > 0$.
At $x = 17$, $\frac{d^2Profit}{dx^2} = 400 - 36 \times 17 < 0$. The optimal $x = 17$.
The optimal output is given by

$$\begin{aligned} y &= 200x + 100x^2 - 3x^3 \\ &= 200 \times 17 + 100 \times 17^2 - 3 \times 17^3 \\ &= 3400 + 28900 - 14739 \\ &= 17561 \end{aligned}$$

Revenue equals to the output price muliplied by the optimal output.

$$revenue = py = 2 \times 17561 = 35122$$

Cost or production equals to the level of input multiplied by the input price.

$$cost = wx = 1998 \times 17 = 33966$$

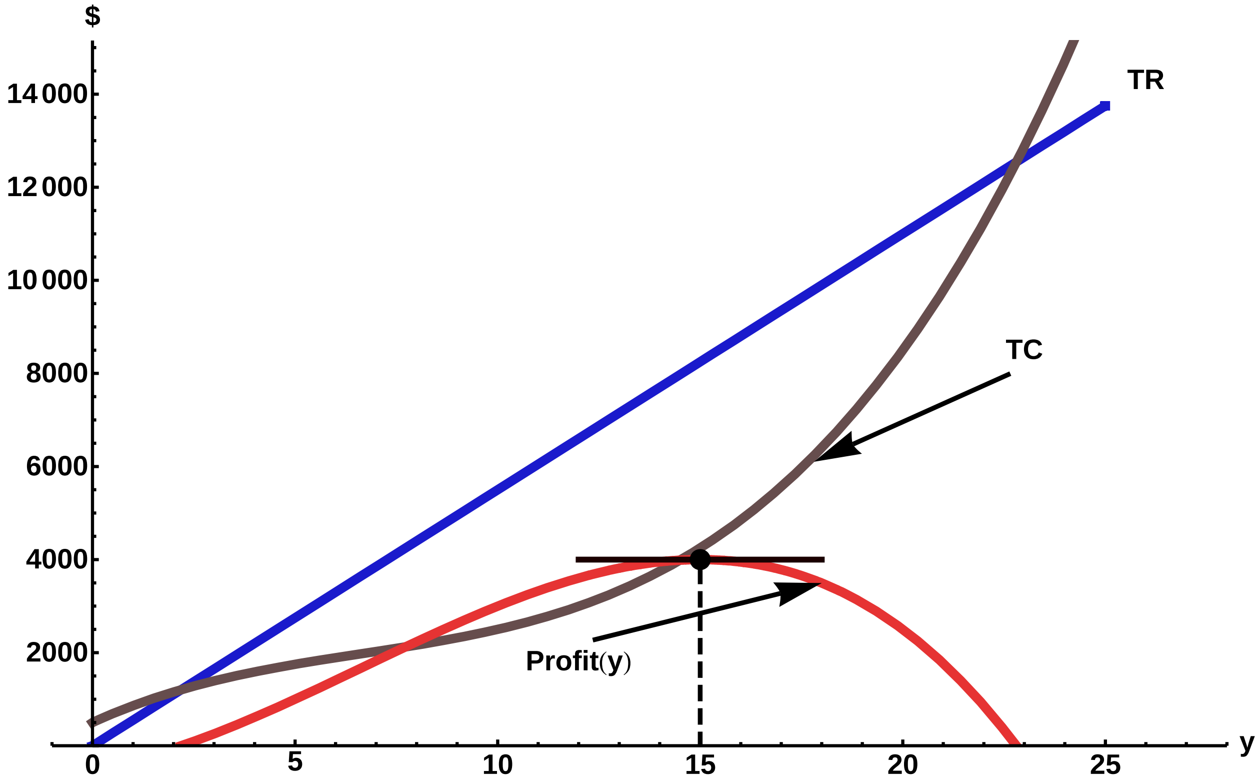
Profit equals to revenue minus cost.

$$profit = revenue - cost = 35122 - 3396 = 1156$$

Problem 4. In this problem (Problem 4), you will be given the price (p) for a competitive firm and the total cost function (TC[y]) for the firm. TC[y] specifies cost as a function of output which is denoted by y. Total revenue is given by TR[y] = py. Marginal cost (MC) is the derivative of the cost function with respect to output. At the optimal level of output, marginal cost will be equal to price.

A graph of total revenue (TR), total cost (TC) and profit for such a firm is given in figure 2. Maximum profit is at the point where total revenue is at the highest level above total cost.

FIGURE 2. Revenue, Cost and Profit



For each problem write down the equation for profit. Then find the profit maximizing level of output.

a.

$$price \;=\; p \;= 550$$
$$Total\ Cost \;=\; TC(y) \;= 500 + 400y - 40y^2 + 2y^3$$

The total revenue is given by

$$Total\ \ revenue = TR(y) = py = 550y$$

The profit is given by

$$Profit = TR(y) - TC(y) = 550y - 500 - 400y + 40y^2 - 2y^3$$
$$= -500 + 150y + 40y^2 - 2y^3$$

The first and second derivative of profit are given by

$$\frac{d\,Profit}{d\,y} = 150 + 80y - 6y^2 \tag{2}$$

$$\frac{d^2\,Profit}{d\,y^2} = 80 - 12y$$

Set the first derivative, equation (2), to zero and solve for x.

$$\frac{d\,Profit}{d\,y} = 150 + 80y - 6y^2 = 0$$

$$\Rightarrow \quad (15 - y)(10 + 6y) = 0$$

$$\Rightarrow \quad y = 15 \quad \text{or} \quad y = -\frac{5}{3}$$

At $y = -\frac{5}{3}$, $\frac{d^2\,Profilt}{d\,y} = 80 - 12 \times (-5/3) > 0$.
At $y = 15$, $\frac{d^2\,Profilt}{d\,y} = 80 - 12 \times 15 = -100$. The optimal $y = 15$.
When the output level $y = 15$, the total profit at the optimal level of output is given by

$$\begin{aligned} Profit &= -500 + 150y + 40y^2 - 2y^3 \\ &= -500 + 150 \times 15 + 40 \times 15^2 - 2 \times 15^3 \\ &= 4000 \end{aligned}$$

When $y = 15$, the total cost at the optimal level of output is given by

$$\begin{aligned} TC(y) &= 500 + 400y - 40y^2 + 2y^3 \\ &= 500 + 400 \times 15 - 40 \times 15^2 + 2 \times 15^3 \\ &= 4250 \end{aligned}$$

And the revenue at the optimal level of output is given by

$$revenue = 550y = 550 \times 15 = 8250.$$

When $y = 15$, The marginal cost at the optimal level of output is given by

$$\begin{aligned} MC(y) &= \frac{d\,(500 + 400y - 40y^2 + 2y^3)}{d\,y} \\ &= 400 - 80y + 6y^2 \\ &= 400 - 80 \times 15 + 6 \times 15^2 \\ &= 550 \end{aligned}$$

So this shows that at the optimal level of output, marginal cost will be equal to price.

b.

$$\begin{aligned} p &= 1164 \\ TC &= 1000 + 800y - 50y^2 + 3y^3 \end{aligned}$$

The total revenue is given by

$$Total\ \ revenue = TR(y) = py = 1164y$$

The profit is given by

$$\begin{aligned} Profit = TR(y) - TC(y) &= 1164y - (1000 + 800y - 50y^2 + 3y^3) \\ &= -1000 + 364y + 50y^2 - 3y^3 \end{aligned}$$

The first and second derivative of profit are given by

$$\frac{d\,Profit}{d\,y} = 364 + 100y - 9y^2 \tag{3}$$

$$\frac{d^2\,Profit}{d\,y^2} = 100 - 18y$$

Set the first derivative, equation (3), to zero and solve for x.

$$\begin{aligned} \frac{d\,Profit}{d\,y} &= 364 + 100y - 9y^2 = 0 \\ \Rightarrow \qquad & (14 - y)(26 + 9y) = 0 \\ \Rightarrow \qquad & y = 14 \quad \text{or } y = -26/9 \end{aligned}$$

When $y = -26/9$, $\frac{d^2\,Profit}{d\,y^2} = 100 - 18 \times (-26/9) = 152 > 0$.
When $y = 14$, $\frac{d^2\,Profit}{d\,y^2} = 100 - 18 \times 14 = -152 < 0$. The optimal $y = 14$.
When the output level $y = 14$, the total profit at the optimal level of output is given by

$$\begin{aligned} Profit &= -1000 + 364y + 50y^2 - 3y^3 \\ &= -1000 + 364 \times 14 + 50 \times 14^2 - 3 \times 14^3 \\ &= -1000 + 364 \times 14 + 50 \times 14^2 - 3 \times 14^3 \\ &= 5664 \end{aligned}$$

When $y = 14$, the total cost at the optimal level of output is given by

$$\begin{aligned} TC(y) &= 1000 + 800y - 50y^2 + 3y^3 \\ &= 1000 + 800 \times 14 - 50 \times 14^2 + 3 \times 14^3 \\ &= 10632. \end{aligned}$$

And the revenue at the optimal level of output is given by

$$revenue = 1164y = 16296.$$

The marginal cost at the optimal level of output is given by

$$\begin{aligned} MC(y) &= \frac{d\,(1000 + 800y - 50y^2 + 3y^3)}{d\,y} \\ &= 800 - 100y + 9y^2 \\ &= 1164 \end{aligned}$$

Problem 5. Solve the following system of equations.

$$324x_1^{-4/5}x_2^{2/5} \;-\; 16 \;= 0$$

$$648x_1^{1/5}x_2^{-3/5} \;-\; 243 \;= 0$$

The x values that solve the system are x_1 = 243 and x_2 =32.

From the second equation,

$$\begin{aligned} & 648x_1^{1/5}x_2^{-3/5} - 243 & ! =! & \;\; 0 \\ \Rightarrow \quad & x_1^{1/5}x_2^{-3/5} & = & \;\; \tfrac{243}{648} \\ \Rightarrow \quad & x_1^{1/5} & = & \;\; \tfrac{3}{8}x_2^{3/5} \\ \Rightarrow \quad & x_1^{-4/5} & = & \;\; \left(\tfrac{8}{3}\right)^4 x_2^{-12/5} \end{aligned}$$

Substitute $x_1^{-4/5} = \left(\frac{8}{3}\right)^4 x_2^{-12/5}$ into the first equation.

$$\begin{aligned} & 324x_1^{-4/5}x_2^{2/5} - 16 &=& \; 0 \\ \Rightarrow \quad & 324\left[\left(\tfrac{8}{3}\right)^4 x_2^{-12/5}\right]x_2^{2/5} &=& \; 16 \\ \Rightarrow \quad & 2^{14}x_2^{-10/5} &=& \; 16 \\ \Rightarrow \quad & x_2^{-10/5} &=& \; 2^{-10} \\ \Rightarrow \quad & x_2 &=& \; 32 \end{aligned}$$

Substitute $x_2 = 32$ into $x_1^{1/5} = \frac{3}{8}x_2^{3/5}$.

$$\begin{aligned} & x_1^{1/5} &=& \; \tfrac{3}{8}x_2^{3/5} \\ \Rightarrow \quad & x_1^{1/5} &=& \; \tfrac{3}{8}32^{3/5} \\ \Rightarrow \quad & x_1^{1/5} &=& \; 3 \\ \Rightarrow \quad & x_1 &=& \; 243 \end{aligned}$$

So the solution is $x_1 = 243,\; x_2 = 32$.

Problem 6. For each of the following, find the points where f has critical values. Check to see whether these are maximum or minimum point.

a. $\{x = 28, x =?\}$
$f(x) = 42x^2 - x^3$

The first and second derivative of $f(x)$ are given by

$$\frac{d\,f(x)}{d\,x} = 84x - 3x^2$$
$$\frac{d^2\,f(x)}{d\,x^2} = 84 - 6x$$

Set the first derivative to be zero. That is,

$$\begin{aligned} 84x - 3x^2 &= 0 \\ \Rightarrow \quad x(28 - x) &= 0 \\ \Rightarrow \quad x = 0 &\text{ or } x = 28 \end{aligned}$$

When $x = 0$, $\frac{d^2\,f(x)}{d\,x^2} = 84 - 6x = 84 > 0$. So $f(x)$ has a minimum point at $x = 0$.

And when $x = 26$, $\frac{d^2\,f(x)}{d\,x^2} = 84 - 6x = -84 < 0$. So $f(x)$ has a maximum point at $x = 28$.

b. $\{x = 26, x =?\}$
$f(x) = -2x^3 + 84x^2 - 312x$

The first and second derivative of $f(x)$ are given by

$$\frac{d\,f(x)}{d\,x} = -6x^2 + 168x - 312$$
$$\frac{d^2\,f(x)}{d\,x^2} = -12x + 168$$

Set the first derivative to be zero. That is,

$$\begin{aligned} -6x^2 + 168x - 312 &= 0 \\ \Rightarrow \quad -6(x - 2)(x - 26) &= 0 \\ \Rightarrow \quad x = 2 &\text{ or } x = 26 \end{aligned}$$

When $x = 2$, $\frac{d^2\,f(x)}{d\,x^2} = -12x + 168 = 144 > 0$. So $f(x)$ has a minimum point at $x = 2$.
And when $x = 26$, $\frac{d^2\,f(x)}{d\,x^2} = -12x + 168 = -144 < 0$. So $f(x)$ has a maximum point at $x = 26$.

c. $f(x) \ = \ -x^3 + 21x^2 + 285x$

The first and second derivative of $f(x)$ are given by

$$\frac{d\ f(x)}{d\ x} = -3x^2 + 42x + 285$$
$$\frac{d^2\ f(x)}{d\ x^2} = -6x + 42$$

Set the first derivative to be zero. That is,

$$\begin{aligned} -3x^2 + 42x + 285 &= 0 \\ \Rightarrow \quad -3(x-19)(x+5) &= 0 \\ \Rightarrow \quad x = -5 \ &\text{or} \ x = 19 \end{aligned}$$

When $x = -5$, $\frac{d^2\ f(x)}{d\ x^2} = -6x + 42 = 72 > 0$. So $f(x)$ has a minimum point at $x = -5$.

And when $x = 19$, $\frac{d^2\ f(x)}{d\ x^2} = -6x + 42 = -72 < 0$. So $f(x)$ has a maximum point at $x = 19$.

d. $f(x) \ = \ -2x^3 + 42x^2 + 192x$

The first and second derivative of $f(x)$ are given by

$$\frac{d\ f(x)}{d\ x} = -6x^2 + 84x + 192$$
$$\frac{d^2\ f(x)}{d\ x^2} = -12x + 84$$

Set the first derivative to be zero. That is,

$$\begin{aligned} -6x^2 + 84x + 192 &= 0 \\ \Rightarrow \quad -6(x-16)(x+2) &= 0 \\ \Rightarrow \quad x = -2 \ &\text{or} \ x = 16 \end{aligned}$$

When $x = -2$, $\frac{d^2\ f(x)}{d\ x^2} = -12x + 84 = 108 > 0$. So $f(x)$ has a minimum point at $x = -2$.

And when $x = 16$, $\frac{d^2\ f(x)}{d\ x^2} = -12x + 84 = -108 < 0$. So $f(x)$ has a maximum point at $x = 16$.

Problem 7. Find the indefinite integral of each of the following functions. Write in the form F(x) + c.

a. $f(x) \ = \ 12x^2 \ + \ 6x \ + \ 3$

b. $f(x) \ = \ 400 - 80x + 6x^2$

c. $y = \frac{12}{\sqrt{x}}$

d. $y = -\frac{7}{3x^{4/3}}$