ECONOMICS 207
SPRING 2008
LABORATORY EXERCISE 7
KEY

Problem 1. Find the second derivative of each of the following functions with respect to x

a. $f(x) \;=\; -400 + 200x + 20x^2 - 2x^3$

$$\begin{aligned}\frac{d\,f(x)}{d\,x} &= 200 + 40x - 6x^2 \\ \frac{d^2\,f(x)}{d\,x^2} &= 40 - 12x\end{aligned}$$

b. $f(x) \;=\; 180x^{1/4}z^{2/5} \;-\; 15x \;-\; 8z$

$$\begin{aligned}\frac{d\,f(x)}{d\,x} &= 180z^{2/5}\left(\frac{1}{4}x^{-3/4}\right) - 15 \\ &= 45z^{2/5}x^{-3/4} - 15 \\ \frac{d^2\,f(x)}{d\,x^2} &= 45z^{2/5}\left(-\frac{3}{4}x^{-7/4}\right) \\ &= -\frac{135}{4}z^{2/5}x^{-7/4}\end{aligned}$$

c. $f(x) \;=\; (2x^3 \;-\; 4x^2)^3$

$$\begin{aligned}\frac{d\,f(x)}{d\,x} &= 3(2x^3 - 4x^2)^2(6x^2 - 8x) \\ \frac{d^2\,f(x)}{d\,x^2} &= 6(2x^3 - 4x^2)(6x^2 - 8x)^2 + 3(2x^3 - 4x^2)^2(12x - 8)\end{aligned}$$

Date: August 6, 2008.

d. $f(x) \; = \; 4x^3 e^{2x^3-4x^2}$

$$
\begin{aligned}
\frac{d\,f(x)}{d\,x} &= 12x^2 e^{2x^3-4x^2} + 4x^3 e^{2x^3-4x^2}(6x^2-8x) \\
&= 12x^2 e^{2x^3-4x^2} + (24x^5-32x^4)e^{2x^3-4x^2} \\
\frac{d^2\,f(x)}{d\,x^2} &= 24x e^{2x^3-4x^2} + 12x^2 e^{2x^3-4x^2}(6x^2-8x) \\
&\qquad + (120x^4-128x^3)e^{2x^3-4x^2} + (24x^5-32x^4)e^{2x^3-4x^2}(6x^2-8x) \\
&= e^{2x^3-4x^2}\left[24x + 12x^2(6x^2-8x) + 120x^4 - 128x^3 + (24x^5-32x^4)(6x^2-8x)\right]
\end{aligned}
$$

Problem 2. Find the definite integral of each of the following functions.

a. $\int_2^4 (6x+3)\ dx$

$$\begin{aligned}\int_2^4 (6x+3)\ dx &= \big(3x^2+3x\big)\Big|_2^4 \\ &= 48+12-(12+6) \\ &= 42\end{aligned}$$

b. $\int_{16}^{81} \big(45x^{-3/4}z^{2/5}\ -\ 15\big)\ dx, \quad z\ =\ 243.$

$$\begin{aligned}\int_{16}^{81} \Big(45x^{-3/4}z^{2/5}\ -\ 15\Big)\ dx &= \left(\frac{45}{-3/4+1}x^{1/4}z^{2/5}-15x\right)\Bigg|_{16}^{81} \\ &= (540z^{2/5}-1215)-(360z^{2/5}-240) \\ &= 180z^{2/5}-975 \\ &= 180\times 9-975 \\ &= 645\end{aligned}$$

c. $\int_0^{10} \left(6x^2 - 40x + 200\right) \, dx$

$$\begin{aligned}\int_0^{10} \left(6x^2 - 40x + 200\right) \, dx &= \left(2x^3 - 20x^2 + 200x\right)\Big|_0^{10} \\ &= 2 \times 1000 - 20 \times 100 + 200 \times 10 \\ &= 2000\end{aligned}$$

d. $\int_0^{12} \left(-27x^2 + 360x - 432\right) \, dx$

$$\begin{aligned}\int_0^{12} \left(-27x^2 + 360x - 432\right) \, dx &= \left(-9x^3 + 180x^2 - 432x\right)\Big|_0^{12} \\ &= -9 \times 12^3 + 120 \times 12^2 - 432 \times 12 \\ &= 5184\end{aligned}$$

Problem 3. Solve the following systems of equations.

$$\begin{aligned} 45x_1^{-3/4}x_2^{2/5} \ - \ 15 \ &= 0 \\ 72x_1^{1/4}x_2^{-3/5} \ - \ 8 \ &= 0 \end{aligned}$$

From the second equation,

$$\begin{aligned} 72x_1^{1/4}x_2^{-3/5} \ - \ 8 &= 0 \\ \Rightarrow \qquad x_1^{1/4}x_2^{-3/5} &= 1/9 \\ \Rightarrow \qquad x_1^{1/4} &= 3^{-2}x_2^{3/5} \\ \Rightarrow \qquad x_1^{-3/4} &= 3^6x_2^{-9/5} \end{aligned}$$

Substitute $x_1^{-3/4} = 3^6x_2^{-9/5}$ into the first equation.

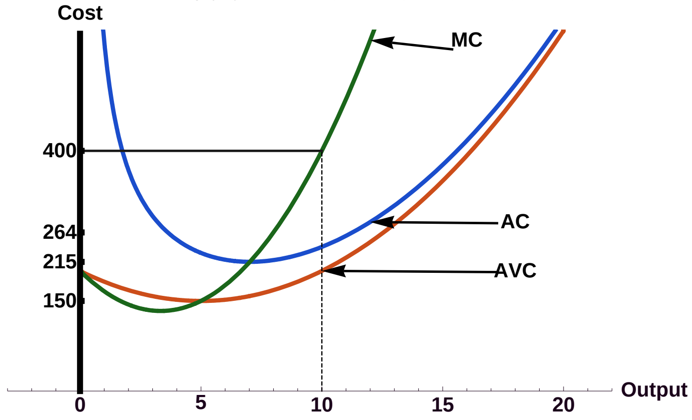
$$\begin{aligned} 45x_1^{-3/4}x_2^{2/5} - 15 &= 0 \\ \Rightarrow \qquad 3^6x_2^{-9/5}x_2^{2/5} &= 3^{-1} \\ \Rightarrow \qquad x_2^{-7/5} &= 3^{-7} \\ \Rightarrow \qquad x_2^{1/5} &= 3 \\ \Rightarrow \qquad x_2 &= 243 \end{aligned}$$

Substitute $x_2 = 243$ into $x_1^{-3/4} = 3^6x_2^{-9/5}$.

$$\begin{aligned} x_1^{-3/4} &= 3^6x_2^{-9/5} \\ \Rightarrow \quad x_1^{-3/4} &= 3^63^{-9} \\ \Rightarrow \quad x_1 &= (3^{-3})^{-4/3} \\ \Rightarrow \quad x_1 &= 3^4 \\ \Rightarrow \quad x_1 &= 81 \end{aligned}$$

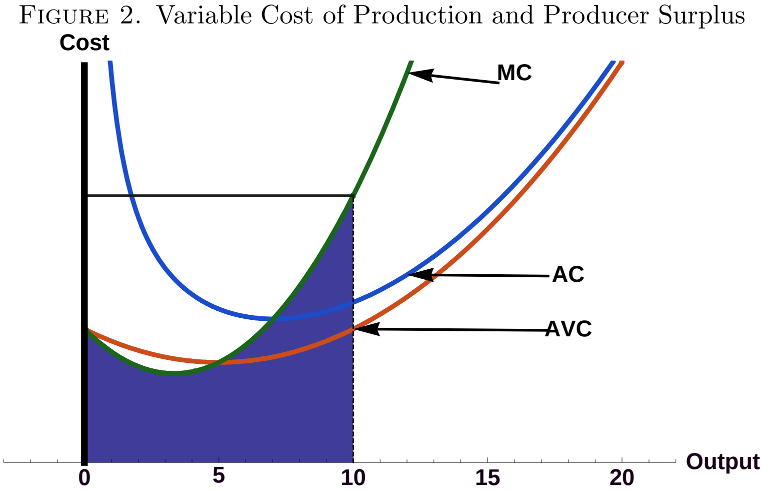
Problem 4. The cost function for a firm is a rule or mapping that tells the total cost of production of any output level produced by the firm. If the variable y represents the output of the firm, then the cost function is given by c(y). Marginal cost represents the change in the cost of production for the firm as output changes and is given by the derivative of the cost function with respect to output,i.e., Marginal Cost(MC) $= \frac{dc(y)}{dy}$. A competitive firm facing a fixed output price maximizes profit at the output level where marginal cost is equal to price as in the figure 1.

FIGURE 1. Profit Maximization



The area below the cost curve is a measure of variable cost and can be found by integrating the marginal cost curve from 0 to any given output level y. The shaded area in figure 2 represents the variable cost of production for the cost function $c(y) = 400 + 200y - 20\,y^2 + 2y^3$.

FIGURE 2. Variable Cost of Production and Producer Surplus



Producer surplus is the area below a given price and above the marginal cost curve. Producer surplus is the unshaded area below the horizontal line at 400 in figure 2. Producer surplus can be computed by subtracting the shaded area from total revenue.

a. Find the profit maximizing level of output for the following firm. Demonstrate that the level you choose maximizes profit.

$$price \;=\; p \;=\$400$$
$$cost \;=\; c(y) \;= 400 \;+\; 200y \;-\; 20y^2 \;+\; 2y^3$$

Given the cost function, $c(y) \;=\; 400 \;+\; 200y \;-\; 20y^2 \;+\; 2y^3$, the profit is given by

$$\begin{aligned} Profit &= py - c(y) \\ &= 400y - (400 + 200y - 20y^2 + 2y^3) \\ &= -2y^3 + 20y^2 + 200y - 400 \end{aligned}$$

The first and seond derivative for the profit function are given by

$$\frac{d\,Profit}{d\,y} = -6y^2 + 40y + 200 \tag{1}$$
$$\frac{d^2\,Profit}{d\,y^2} = -12y + 40$$

Let the first derivative of profit be zero, (1) = 0. That is,

$$\begin{aligned} 200 + 40y - 6y^2 &= 0 \\ \Rightarrow \quad 6y^2 - 40y - 200 &= 0 \\ \Rightarrow \quad (y-10)(6y+20) &= 0 \\ \Rightarrow \quad y = 10 \quad \text{or} \quad y = -\tfrac{10}{3} \end{aligned}$$

When $y = -\frac{10}{3}$, $\frac{d^2\,Profit}{d\,y^2} = -12y + 40 = 80 > 0$.
And When $y = 10$, $\frac{d^2\,Profit}{d\,y^2} = -12y + 40 = -80 < 0$.
So the optimal ouput for profit maximization is that $y = 10$,

b. What is revenue minus variable cost for this firm when price is $400?

At the optimal output level of maximizing profit,

$$Revenue = py = \$400 \times 10 = \$4000$$

And the variable cost is given by

$$200y - 20y^2 + 2y^3 = \$(2000 - 2000 + 2000) = \$2000$$

So the revenue minus variable cost is given by

$$Revenue \ - variable\ cost = \$4000 - \$2000 = \$2000$$

c. Find producer surplus for this firm assuming you are only given the following marginal cost function: $MC(y) = 200 - 40y + 6y^2$ and a price of $400.

The producer surplus is given by

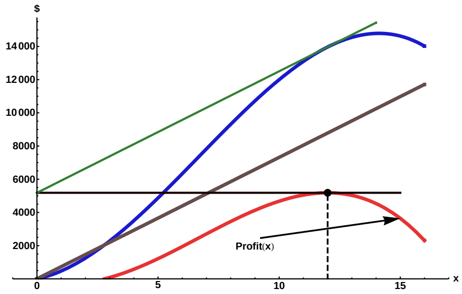
$$\begin{aligned}
producer\ surplus &= total\ revenue - \int_0^{10} MC(y)dy \\
&= 400 \times 10 - \int_0^{10} (200 - 40y + 6y^2)dy \\
&= 4000 - \left(200y - 20y^2 + 2y^3\right)\Big|_0^{10} \\
&= 4000 - 2000 \\
&= 2000
\end{aligned}$$

Problem 5. In the following problem you are given a production function for a firm where y is the level of output and x is the level of the variable input. You are given the price (p) of the output and the price (w) of the single variable input.

a. Write down an equation that represents profit for the firm.
b. Maximize this function by taking its derivative with respect to the variable input x and setting the derivative equal to zero.
c. Solve the equation in part 5b for the potentially profit maximizing level of x.
d. Determine using the second order conditions which of the roots represents maximum profit.

$$\begin{aligned} output\ price &= p = 3 \\ input\ price &= w = 732 \\ y = output &= f(x) = 100x + 60x^2 - 3x^3 \end{aligned}$$

FIGURE 3. Revenue, Cost and Profit



a. The profit for the firm is given by

$$\begin{aligned} Profit &= py - wx \\ &= 3(100x + 60x^2 - 3x^3) - 732x \\ &= -9x^3 + 180x^2 - 432x \end{aligned}$$

b. the first derivative of the profit function is given by

$$\frac{d\,(-9x^3 + 180x^2 - 432x)}{d\,x} = -27x^2 + 360x - 432 \tag{2}$$

Set equation (2) to zero.

$$-27x^2 + 360x - 432 = 0$$

c.

$$\begin{aligned} -27x^2 + 360x - 432 &= 0 \\ \Rightarrow \quad -9(x - 12)(3x - 4) &= 0 \\ \Rightarrow \quad x = 12 \ \text{ or } \ x &= \tfrac{4}{3} \end{aligned}$$

d. The second derivative of the profit function is given by

$$\frac{d^2\left(-9x^3+180x^2-432x\right)}{d\,x^2} = -54x+360 \tag{3}$$

When $x=\frac{4}{3}$, the second derivative of the profit, equation (3), is given by

$$-54x+360 = -54\times\frac{4}{3}+360 = 288 > 0$$

When $x=12$, the second derivative of the profit, equation (3), is given by

$$-54x+360 = -54\times 12+360 = -288 < 0$$

So the profit attains maximum when the input level $x=12$.

Problem 6. Solve the following system of equations for x_1, x_2, and x_3.

$$\{x_1 \;=\; 3,\; x_2 \;=\; 2,\; x_3 \;=\; 1\}$$

$$\begin{aligned} x_1 \;+\; 2x_2 \;-\; 3x_3 \;&= 4 \\ -3x_1 \;-\; 5x_2 \;+\; 8x_3 \;&= -11 \\ 2x_1 \;+\; 5x_2 \;-\; 6x_3 \;&= 10 \end{aligned}$$

Multiply the first equation by 3 and add it to the second equation.

$$3x_1 + 6x_2 - 9x_3 + (-3x_1 - 5x_2 + 8x_3) = 4 \times 3 - 11$$

That is,

$$x_2 - x_3 = 1 \tag{4}$$

Multiply the first equation by -2 and add it the third equation.

$$-2x_1 - 4x_2 + 6x_3 + (2x_1 + 5x_2 - 6x_3) = 4 \times (-2) + 10$$

That is,

$$x_2 = 2 \tag{5}$$

Substitute $x_2 = 2$ into $x_2 - x_3 = 1$, equation (4).

$$\begin{aligned} x_2 - x_3 &= 1 \\ \Rightarrow \quad 2 - x_3 &= 1 \\ \Rightarrow \quad x_3 &= 1 \end{aligned}$$

Substitute $x_2 = 2$ and $x_3 = 1$ into the first equation.

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 4 \\ \Rightarrow \quad x_1 + 4 - 3 &= 4 \\ \Rightarrow \quad x_1 &= 3 \end{aligned}$$

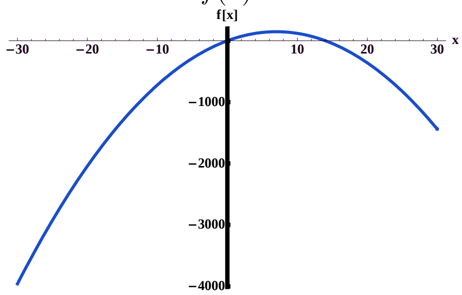
So the solution is

$$x_1 = 3,\; x_2 = 2,\; x_3 = 1$$

Problem 7. For each of the following problems, find the critical points. For each critical point state whether the function is at a relative maximum, relative minimum, or otherwise.

a. $y = 42x - 3x^2$

Figure 4. $f(x) = 42x - 3x^2$



The first and second derivative of $y = 42x - 3x^2$ are given by

$$\frac{d\,y}{d\,x} = 42 - 6x \tag{6}$$

$$\frac{d^2\,y}{d\,x^2} = -6 \tag{7}$$

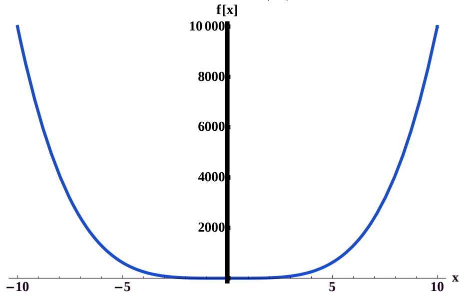
Set equation (6) to zero.

$$42 - 6x = 0 \quad \Rightarrow \quad x = 7$$

When $x = 7$, the second derivative, equation (7), is -6. So $x = 7$ is a point where $y = 42x - 3x^2$ attains its global maximum.

b. $y = x^4$

FIGURE 5. $f(x) = x^4$



The derivatives of $y = x^4$ are given by

$$\frac{d\,y}{d\,x} = 4x^3 \tag{8}$$

$$\frac{d^2\,y}{d\,x^2} = 12x^2 \tag{9}$$

$$\frac{d^3\,y}{d\,x^3} = 24x \tag{10}$$

$$\frac{d^4\,y}{d\,x^4} = 24 \tag{11}$$

Set the first derivative, equation (8), to zero.

$$4x^3 = 0 \quad \Rightarrow \quad x = 0$$

When $x = 0$, the second [equation (9)], the third [equation (10)], and the fourth [equation (11)] derivatives are given by

$$\frac{d^2\,y}{d\,x^2} = 12x^2 = 0$$

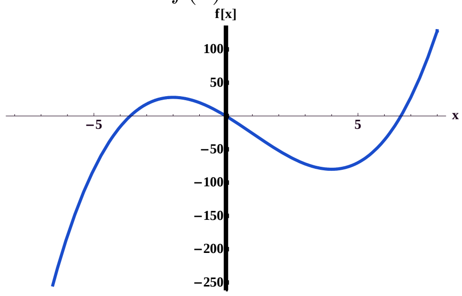
$$\frac{d^3\,y}{d\,x^3} = 24x = 0$$

$$\frac{d^4\,y}{d\,x^4} = 24 > 0$$

So $x = 0$ is a point where $y = x^4$ attains its global minimum.

c. $f(x) = x^3 - 3x^2 - 24x$

FIGURE 6. $f(x) = x^3 - 3x^2 - 24x$



The derivatives of $y = x^3 - 3x^2 - 24x$ are givne by

$$\frac{d\,y}{d\,x} = 3x^2 - 6x - 24 \tag{12}$$

$$\frac{d^2\,y}{d\,x^2} = 6x - 6 \tag{13}$$

$$\frac{d^3\,y}{d\,x^3} = 6 \tag{14}$$

Set the first derivative, equation (12), to zero.

$$\begin{aligned} 3x^2 - 6x - 24 &= 0 \\ \Rightarrow \quad 3(x-4)(x+2) &= 0 \\ \Rightarrow \quad x = 4 &\text{ or } x = -2 \end{aligned}$$

When $x = -2$, the second derivative, equation (13), is given by

$$\frac{d^2\,y}{d\,x^2} = 6x - 6 = -18 < 0$$

So $x = -2$ is a point where the function is at a relative maximum.

When $x = 4$, the second derivative, equation (13), is given by

$$\frac{d^2\,y}{d\,x^2} = 6x - 6 = 18 > 0$$

So $x = 4$ is a point where the function is at a relative minimum.

Set the second derivative, equation (13), to zero.

$$\frac{d^2\,y}{d\,x^2} = 6x - 6 = 0 \quad \Rightarrow \quad x = 1$$

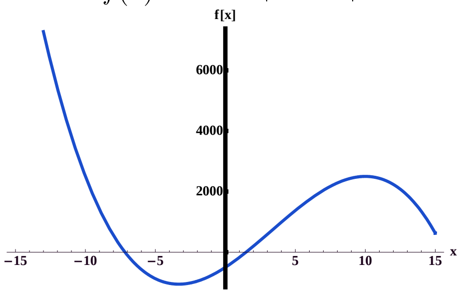
When $x = 1$, the third derivative, equation (14), is given by

$$\frac{d^3\,y}{d\,x^3} = 6 \neq 0$$

So $x = 1$ is an inflection point of the function.

d. $f(x) = -3x^3 + 30x^2 + 300x - 500$

FIGURE 7. $f(x) = -3x^3 + 30x^2 + 300x - 500$



The derivatives of $f(x) = -3x^3 + 30x^2 + 300x - 500$ are given by

$$f'(x) = -9x^2 + 60x + 300 \tag{15}$$

$$f''(x) = -18x + 60 \tag{16}$$

$$f^{(3)} = -18 \tag{17}$$

Set the first derivative, equation (15), to zero.

$$\begin{aligned} f'(x) = -9x^2 + 60x + 300 &= 0 \\ \Rightarrow \qquad -3(x-10)(3x+10) &= 0 \\ \Rightarrow \qquad x = 10 \quad &\text{or} \quad x = -\tfrac{10}{3} \end{aligned}$$

When $x = 10$, the second derivative, equation (16), is given by

$$f''(10) = -18 \times 10 + 60 = -120 < 0$$

So $x = 10$ is a point where the function is at a relative maximum.

When $x = -\frac{10}{3}$, the second derivative, equation (16), is given by

$$f''(-10/3) = -18\left(-\frac{10}{3}\right) + 60 = 120 > 0$$

So $x = -10/3$ is a point where the function is at a relative minimum.

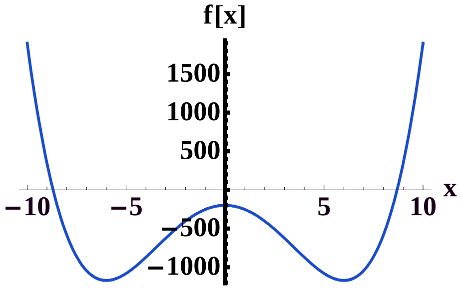
Set the second derivative, equation (16), to zero.

$$f''(x) = -18x + 60 = 0 \quad \Rightarrow \quad x = \frac{10}{3}$$

When $x = \dfrac{10}{3}$, the third derivative, equation (17), is -18, which is not equal to zero. So $x = \dfrac{10}{3}$ is an inflection point of the function.

e. $f(x) = 3/4x^4 - 54x^2 - 200$

FIGURE 8. $f(x) = 3/4x^4 - 54x^2 - 200$



The derivatives of $f(x) = \frac{3}{4}x^4 - 54x^2 - 200$ are given by

$$f'(x) = 3x^3 - 108x \tag{18}$$

$$f''(x) = 9x^2 - 108 \tag{19}$$

$$f^{(3)}(x) = 18x \tag{20}$$

Set the first derivative, equation (18), to zero.

$$\begin{aligned} f'(x) = 3x^3 - 108x &= 0 \\ \Rightarrow \qquad 3x(x^2 - 36) &= 0 \\ \Rightarrow \qquad 3x(x+6)(x-6) &= 0 \\ \Rightarrow \qquad x = 0 \quad \text{or} \quad x = 6 \quad &\text{or} \quad x = -6 \end{aligned}$$

When $x = 0$, the second derivative, equation (19), is given by

$$f''(x) = 9 \times 0^2 - 108 = -108 < 0$$

So $x = 0$ is a point where the function is at a relative maximum.

When $x = \pm 6$, the second derivative, equation (19), is given by

$$f''(\pm 6) = 9 \times (\pm 6)^2 - 108 = 216 > 0$$

In addition, $f(6) = f(-6)$. So $x = \pm 6$ are points where the function is at a global minimum.

Set the second derivative, equation (19), to zero.

$$\begin{aligned} f''(x) = 9x^2 - 108 &= 0 \\ \Rightarrow \qquad x^2 &= 12 \\ \Rightarrow \qquad x = 2\sqrt{3} \quad &\text{or} \quad x = -2\sqrt{3} \end{aligned}$$

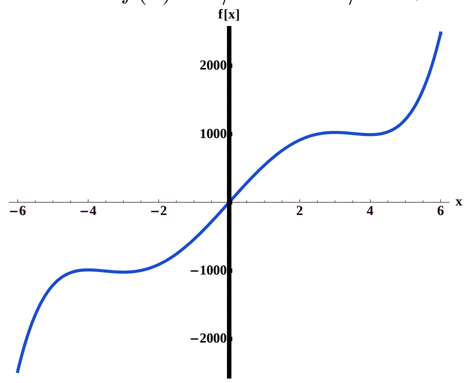
When $x = \pm 2\sqrt{3}$, the third derivative, equation (20), is given by

$$f^{(3)}(\pm 2\sqrt{3}) = 18x = \pm 36\sqrt{3} \neq 0$$

So $x = \pm 2\sqrt{3}$ are inflection points of the function.

f. $f(x) = 4/5x^5 - 100/3x^3 + 576x$

FIGURE 9. $f(x) = 4/5x^5 - 100/3x^3 + 576x$



The derivatives of $f(x) = \frac{4}{5}x^5 - \frac{100}{3}x^3 + 576x$ are given by

$$f'(x) = 4x^4 - 100x^2 + 576 \tag{21}$$

$$f''(x) = 16x^3 - 200x \tag{22}$$

$$f^{(3)}(x) = 48x^2 - 200 \tag{23}$$

Set the first derivative, equation (21), to zero.

$$\begin{aligned}
f'(x) = 4x^4 - 100x^2 + 576 &= 0 \\
\Rightarrow \quad x^4 - 25x^2 + 144 &= 0 \\
\Rightarrow \quad x^4 - 25x^2 + 144 &= 0 \\
\Rightarrow \quad (x^2 - 9)(x^2 - 16) &= 0 \\
\Rightarrow \quad x = 3 \quad \text{or} \quad x = -3 \quad \text{or} \quad x = 4 \quad \text{or} \quad x = -4
\end{aligned}$$

When $x = \pm 3$ and $x = \pm 4$, the second derivative, equation (22), are respectively given by

$$f''(3) = 16 \times 3^3 - 200 \times 3 = -168 < 0$$

$$f''(-3) = 16 \times (-3)^3 - 200 \times (-3) = 168 > 0$$

$$f''(4) = 16 \times 4^3 - 200 \times 4 = 224 > 0$$

$$f''(-4) = 16 \times (-4)^3 - 200 \times (-4) = -224 < 0$$

So $x = 3$ and $x = -4$ are points where the function is at a relative maximum. And $x = -3$ and $x = 4$ are points where the function is at a relative minimum.

Set the second derivative, equation (22), to zero.

$$\begin{aligned}
16x^3 - 200x &= 0 \\
\Rightarrow \quad 4x(4x^2 - 50) &= 0 \\
\Rightarrow \quad x = 0 \quad \text{or} \quad x = \tfrac{5}{\sqrt{2}} \quad &\text{or} \quad x = -\tfrac{5}{\sqrt{2}}
\end{aligned}$$

When $x = 0, \frac{5}{\sqrt{2}},$ and $-\frac{5}{\sqrt{2}}$, the third derivative, equation (23), are respectively given by

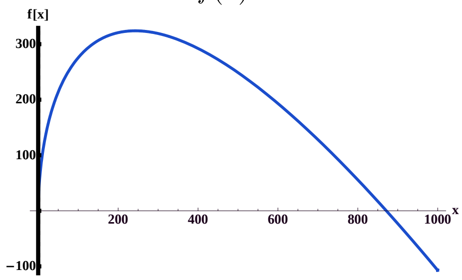
$$f^{(3)}(0) = -200 \neq 0$$

$$f^{(3)}(\pm\frac{5}{\sqrt{2}}) = 48 \times \left(\pm\frac{5}{\sqrt{2}}\right)^2 - 200 = 400 \neq 0$$

So $x = 0$, $x = \frac{5}{\sqrt{2}}$, and $x = -\frac{5}{\sqrt{2}}$, are inflection points of the function.

g. $f(x) = 30x^{3/5} - 2x$

FIGURE 10. $f(x) = 30x^{3/5} - 2x$



The derivatives are of $f(x) = 30x^{3/5} - 2x$ are given by

$$f'(x) = 18x^{-2/5} - 2 \tag{24}$$

$$f''(x) = -\frac{36}{5}x^{-7/5} \tag{25}$$

Set the first derivative, equation (24), to zero.

$$\begin{aligned} f'(x) = 18x^{-2/5} - 2 &= 0 \\ \Rightarrow \quad x^{-2/5} &= 2/9 = 3^{-2} \\ \Rightarrow \quad x^{2/5} &= 3^2 \\ \Rightarrow \quad x &= 3^5 = 243 \end{aligned}$$

When $x \in (0, \infty)$, the second derivative, equation (25), is less than zero. So $x = 243$ is a point where the function is at a global maximum. And there is no inflection point of this function.