ECONOMICS 207
SPRING 2008
LABORATORY EXERCISE 8

Problem 1. Consider the following matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 4 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

$$F = \begin{bmatrix} -2 & -2 & 4 \\ -3 & -1 & 4 \\ -6 & -6 & 10 \end{bmatrix} \qquad G = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 9 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \qquad c = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Compute the following

a. A + C

b. A + 2C

c. A + B

Date: March 5, 2008.

d. AC

e. a′

f. a′b

g. CA

h. Ba

i. Bb

j. $c'a$

k. $c'B$

l. FG

m. Fa

n. $b'G$

o. $A + 2D$

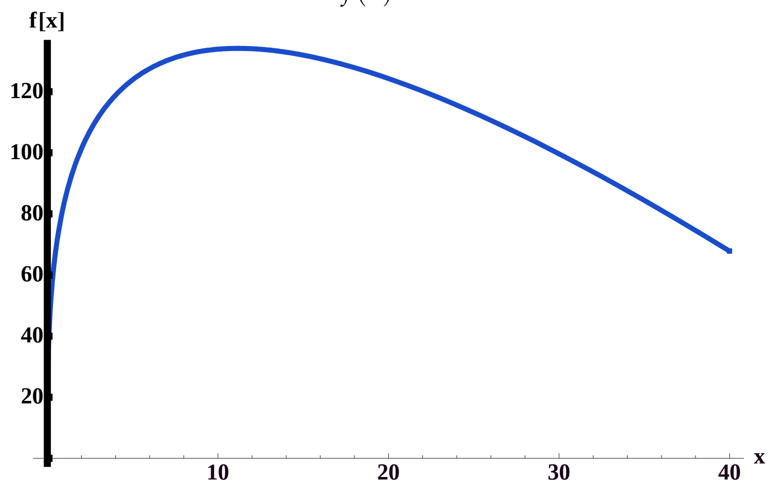
p. $B'D$

q. BG

Problem 2. For each of the following problems, find the critical points. For each critical point state whether the function is at a relative maximum, relative minimum, or otherwise. Also find the points of inflection for each function.

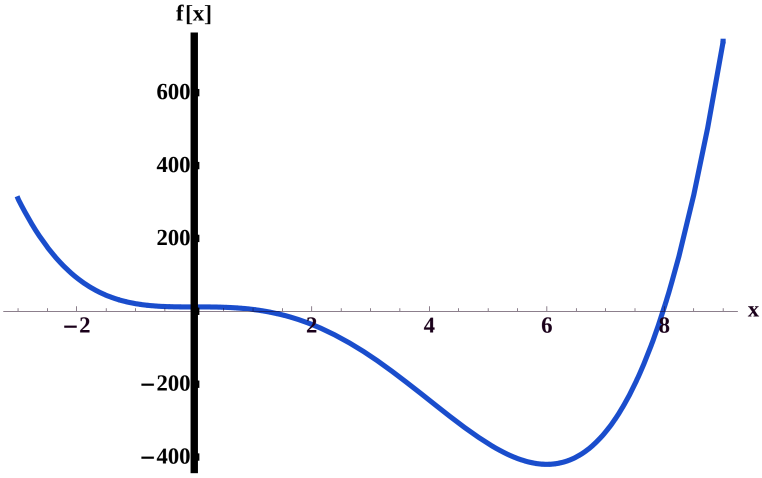
a. $f(x) = 90x^{1/3} - 6x$

FIGURE 1. $f(x) = 90x^{1/3} - 6x$



b. $f(x) = x^4 - 8x^3 + 12$

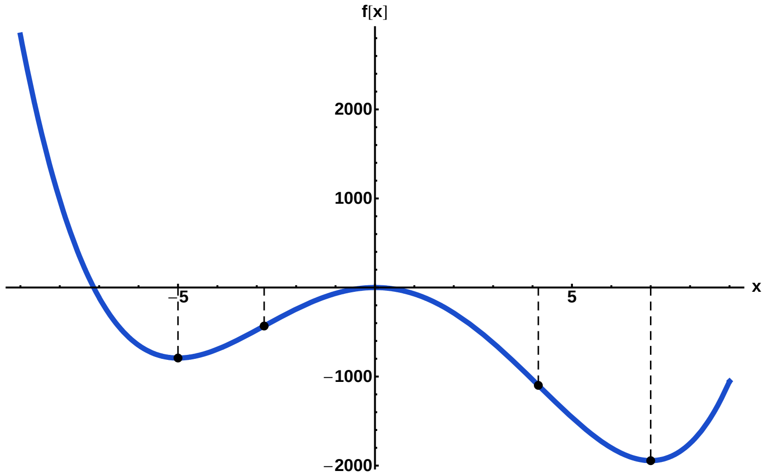
FIGURE 2. $f(x) = x^4 - 8x^3 + 12$



c. $f(x) = x^4 - \frac{8}{3}x^3 - 70x^2$

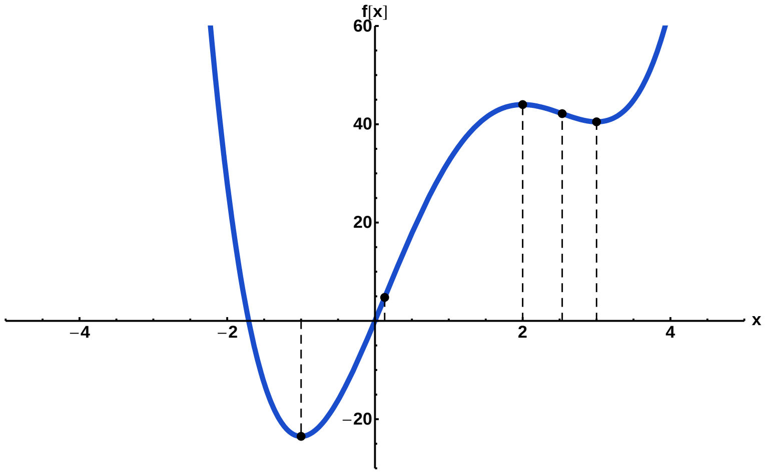
Hint: The inflection points are $x = \frac{2 \pm \sqrt{109}}{3}$

FIGURE 3. $f(x) = x^4 - \frac{8}{3}x^3 - 70x^2$



d. $f(x) = \frac{3}{2}x^4 - 8x^3 + 3x^2 + 36x$
Hint: The inflection points are $x = \frac{4 \pm \sqrt{13}}{3}$

FIGURE 4. $f(x) = \frac{3}{2}x^4 - 8x^3 + 3x^2 + 36x$



Find the first and second derivatives of the function.

$$\begin{aligned} f'(x) &= 6x^3 - 24x^2 + 6x + 36 \\ f''(x) &= 18x^2 - 48x + 6 \end{aligned}$$

Set the first derivative equal to zero.

$$f'(x) = 6x^3 - 24x^2 + 6x + 36 = 0$$

If there are integer roots to this equation,we can factor the cubic if we know one of them. We can guess that x = -1 is a root of this equation so we know that we must have $(x+1)(x+a)(x+b) = 6x^3 - 24x^2 + 6x + 36 = 0$. So if we divide $6x^3 - 24x^2 + 6x + 36$ by (x+1) we can obtain a quadratic represented by (x+a)(x+b). Hopefully we can factor this expression. So let's begin.

If we divide $6x^3 - 24x^2 + 6x + 36$ by (x+1) we first obtain

$$\begin{array}{r|l} & 6x^2 \\ \hline x+1) & 6x^3 - 24x^2 \quad + 6x + 36 \\ & -6x^3 \quad - 6x^2 \\ \hline \end{array}$$

$$
\begin{array}{rl}
 & 6x^2 \\ \cline{2-2}
x+1) & 6x^3 - 24x^2 + 6x + 36 \\
 & -6x^3 - 6x^2 \\ \cline{2-2}
 & - 30x^2 + 6x
\end{array}
$$

Continuing we obtain

$$
\begin{array}{rl}
 & 6x^2 - 30x \\ \cline{2-2}
x+1) & 6x^3 - 24x^2 + 6x + 36 \\
 & -6x^3 - 6x^2 \\ \cline{2-2}
 & - 30x^2 + 6x \\
 & 30x^2 + 30x \\ \cline{2-2}
\end{array}
$$

and then

$$
\begin{array}{rl}
 & 6x^2 - 30x \\ \cline{2-2}
x+1) & 6x^3 - 24x^2 + 6x + 36 \\
 & -6x^3 - 6x^2 \\ \cline{2-2}
 & - 30x^2 + 6x \\
 & 30x^2 + 30x \\ \cline{2-2}
 & 36x + 36
\end{array}
$$

$$
\begin{array}{rl}
 & 6x^2 - 30x + 36 \\ \cline{2-2}
x+1) & 6x^3 - 24x^2 + 6x + 36 \\
 & -6x^3 - 6x^2 \\ \cline{2-2}
 & - 30x^2 + 6x \\
 & 30x^2 + 30x \\ \cline{2-2}
 & 36x + 36
\end{array}
$$

$$\begin{array}{r} 6x^2 - 30x + 36 \\ x+1 \overline{\smash{\big)}\ 6x^3 - 24x^2 \ + 6x + 36} \\ \underline{-6x^3 \ - 6x^2} \qquad\qquad \\ -30x^2 \ + 6x \qquad \\ \underline{30x^2 + 30x} \qquad \\ 36x + 36 \\ \underline{-36x - 36} \\ 0 \end{array}$$

We can now factor the remaining term as follows

$$\begin{aligned} 6x^2 - 30x + 36 &= 6(x^2 - 5x + 6) \\ &= 6(x-3)(x-2) \end{aligned}$$

We then have

$$\begin{aligned} (x+1)6(x-3)(x-2) &= 0 \\ \Rightarrow x = -1, x = 3, x = 2 \end{aligned}$$

We can check each of the roots in the second derivative.

$$\begin{aligned} f''(-1) &= 18(-1)^2 - 48(-1) + 6 \\ &= 18 + 48 + 6 \ = \ 72 \end{aligned}$$

So x = -1 is a minimum point.

$$\begin{aligned} f''(2) &= 18(2)^2 - 48(2) + 6 \\ &= 72 - 96 + 6 \ = \ -18 \end{aligned}$$

So x = 2 is a maximum point.

$$\begin{aligned} f''(3) &= 18(3)^2 - 48(3) + 6 \\ &= (18)(9) - (48)(3) + 6 \\ &= 162 - 144 + 6 \ = \ 24 \end{aligned}$$

So x = 3 is a minimum point.
To find the points of inflection we set the second derivative equal to zero.

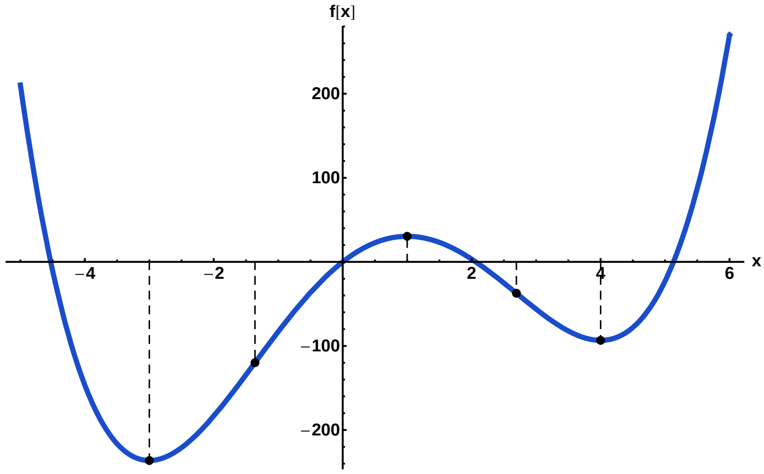
$$\begin{aligned} f''(x) &= 18(x)^2 - 48(x) + 6 \ = \ 0 \\ &= 6(3x^2 - 8x + 1) \ = \ 0 \end{aligned}$$

And unfortunately it does not factor so we obtain

$$
\begin{aligned}
x &= \frac{8 \pm \sqrt{64 - (4)(3)(1)}}{6} \\
&= \frac{8 \pm \sqrt{52}}{6} \\
&= \frac{8 \pm 2\sqrt{13}}{6} \\
&= \frac{4 \pm \sqrt{13}}{3}
\end{aligned}
$$

e. $f(x) = \frac{5}{4}x^4 - \frac{10}{3}x^3 - \frac{55}{2}x^2 + 60x$
Hint: The inflection points are $x = \frac{2 \pm \sqrt{37}}{3}$

FIGURE 5. $f(x) = \frac{5}{4}x^4 - \frac{10}{3}x^3 - \frac{55}{2}x^2 + 60x$



Problem 3. Find the definite integral of each of the following functions.

a. $\int_1^3 \left(4x^3 - 3x^2 - 5x\right) dx$

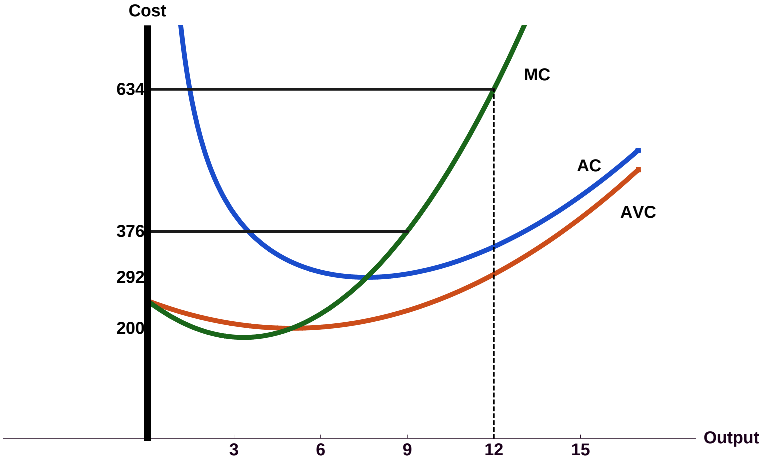
b. $\int_2^6 e^{4x}\, dx$

Problem 4. Solve the following systems of equations.

$$\begin{aligned} 81x_1^{-4/5}x_2^{2/3} - 25 &= 0 \\ 270x_1^{1/5}x_2^{-1/3} - 162 &= 0 \end{aligned}$$

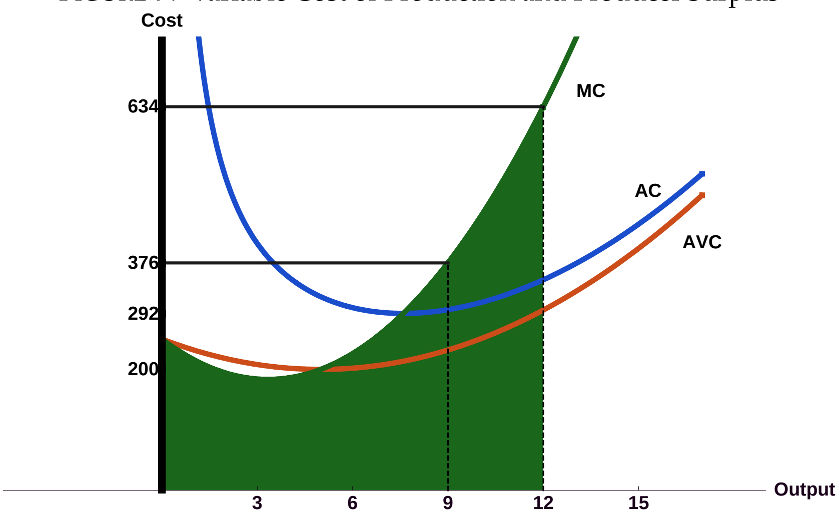
Problem 5. The cost function for a firm is a rule or mapping that tells the total cost of production of any output level produced by the firm. If the variable y represents the output of the firm, then the cost function is given by c(y). Marginal cost represents the change in the cost of production for the firm as output changes and is given by the derivative of the cost function with respect to output,i.e., Marginal Cost(MC) = $\frac{dc(y)}{dy}$. A competitive firm facing a fixed output price maximizes profit at the output level where marginal cost is equal to price as in the figure 6.

FIGURE 6. Profit Maximization



The area below the cost curve is a measure of variable cost and can be found by integrating the marginal cost curve from 0 to any given output level y. The shaded area in figure 7 represents the variable cost of production for the cost function $c(y) = 600 + 250y - 20\,y^2 + 2y^3$.

FIGURE 7. Variable Cost of Production and Producer Surplus



Producer surplus is the area below a given price and above the marginal cost curve. Producer surplus is the unshaded area below the horizontal line at 634 in figure 7. Producer surplus can be computed by subtracting the shaded area from total revenue.

a. Find the profit maximizing level of output for the following firm. Demonstrate that the level you choose maximizes profit.

$$\begin{aligned} price &= p &&= \$634 \\ cost &= c(y) &&= 600 + 250y - 20y^2 + 2y^3 \end{aligned}$$

b. Find the profit maximizing level of output when the price is \$376. Demonstrate that the level you choose maximizes profit.

c. Explain in words why setting price equal to marginal cost and solving for the optimal output y gives the same answers as taking the derivative of profit with respect to y, setting the result equal to zero and solving for the optimal y. Remember that

$$
\begin{aligned}
Profit &= py - c(y) \\
Profit &= 634y - \left[600 + 250y - 20\,y^2 + 2y^3\right]
\end{aligned}
$$

d. What is variable cost for this firm when it maximizes profit with a price of $634?

e. What is producer surplus for this profit maximizing firm when the price is $634?

f. What is variable cost for this firm when when it maximizes profit with a price of $376?

g. What is producer surplus for this profit maximizing firm when the price is $376?

h. How much is the firm worse off when price falls from $634 to $376?

i. Cross-hatch the change in producer surplus in Figure 7.

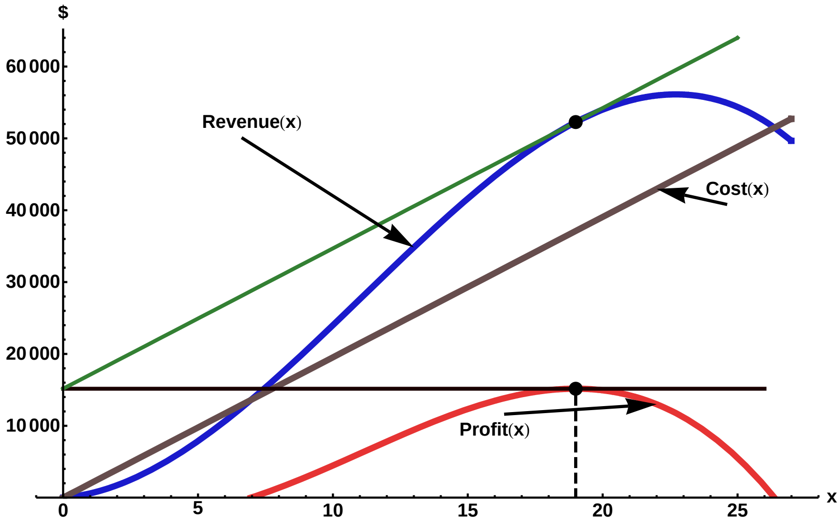
Problem 6. In the following problem you are given a production function for a firm where y is the level of output and x is the level of the variable input. You are given the price (p) of the output and the price (w) of the single variable input.

$$output\ price = p = 3$$

$$input\ price = w = 1953$$

$$y = output = f(x) = 100x + 100x^2 - 3x^3$$

FIGURE 8. Revenue, Cost and Profit



a. Write down an equation that represents profit for the firm.

b. Maximize this function by taking its derivative with respect to the variable input x and setting the resulting equation equal to zero.

c. If you identify more than one critical value from setting the first derivative of profit equal to zero, show which ones, if any, maximize profit.

d. What is the optimal level of output for this firm?

e. Explain in words why the slope of the total revenue function for this firm is equal to the price of the single variable input at the profit maximizing level of input use. You can use the graph 8 and the following information in explaining this phenomenon.

$$\begin{aligned} Revenue &= pf(x) \\ Cost &= wx \\ Profit &= Revenue - Cost = pf(x) - wx \end{aligned}$$

Problem 7.

You do not need to do this problem. Remove it from the packet and use for practice.

Solve the following system of equations for x_1, x_2, and x_3 using the method of elimination.

$$\{x_1 = 1,\ x_2 = -1,\ x_3 = 3\}$$

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= 6 \\ -2x_1 - 9x_2 - 8x_3 &= -17 \\ -3x_1 - 8x_2 + 0x_3 &= 5 \end{aligned}$$