ECONOMICS 207
SPRING 2008
LABORATORY EXERCISE 8
KEY

Problem 1. Consider the following matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 4 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

$$F = \begin{bmatrix} -2 & -2 & 4 \\ -3 & -1 & 4 \\ -6 & -6 & 10 \end{bmatrix} \qquad G = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 9 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \qquad c = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Compute the following

a. A + C

$$A + C = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$$

b. A + 2C

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 6 & 5 \end{bmatrix}$$

c. A + B

The matrix addition is defined for two matrices of the same dimensions, so $A+B$ cannot be added.

Date: August 6, 2008.

d. AC

$$AC = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 2 \times 2 & 1 \times 0 + 2 \times 1 \\ 2 \times 1 + 3 \times 2 & 2 \times 0 + 3 \times 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 8 & 3 \end{bmatrix}$$

e. a'

$$a' = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}' = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

f. a'b

$$\begin{aligned} a'b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}' \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} &= \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \times 0 + 2 \times 3 + 3 \times 1 \end{bmatrix} \\ &= [9] \end{aligned}$$

g. CA

$$CA = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 0 \times 2 & 1 \times 2 + 0 \times 3 \\ 2 \times 1 + 1 \times 2 & 2 \times 2 + 1 \times 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}$$

h. Ba

$$Ba = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \times 1 + 1 \times 2 + 6 \times 3 \\ 3 \times 1 + 4 \times 2 + 5 \times 3 \end{bmatrix} = \begin{bmatrix} 22 \\ 26 \end{bmatrix}$$

i. Bb

$$Bb = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \times 0 + 1 \times 3 + 6 \times 1 \\ 3 \times 0 + 4 \times 3 + 5 \times 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 17 \end{bmatrix}$$

j. c'a

The dimension of matrix c' is 1×2, and the dimension of matrix a is 3×1. So the multiplication of $c'a$ is not defined.

k. c'B

$$\begin{aligned} c'B &= \begin{bmatrix} 1 \\ 4 \end{bmatrix}' \begin{bmatrix} 2 & 1 & 6 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & 6 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 \times 2 + 4 \times 3 & 1 \times 1 + 4 \times 4 & 1 \times 6 + 4 \times 5 \end{bmatrix} \\ &= \begin{bmatrix} 14 & 17 & 26 \end{bmatrix} \end{aligned}$$

l. FG

$$\begin{aligned} FG &= \begin{bmatrix} -2 & -2 & 4 \\ -3 & -1 & 4 \\ -6 & -6 & 10 \end{bmatrix} \begin{bmatrix} 4 & 2 & 0 \\ 2 & 9 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -2 \times 4 + (-2) \times 2 + 4 \times 0 & -2 \times 2 + (-2) \times 9 + 4 \times 0 & -2 \times 0 + (-2) \times 0 + 4 \times 2 \\ -3 \times 4 + (-1) \times 2 + 4 \times 0 & -3 \times 2 + (-1) \times 9 + 4 \times 0 & -3 \times 0 + (-1) \times 0 + 4 \times 2 \\ -6 \times 4 + (-6) \times 2 + 10 \times 0 & -6 \times 2 + (-6) \times 9 + 10 \times 0 & -6 \times 0 + (-6) \times 0 + 10 \times 2 \end{bmatrix} \\ &= \begin{bmatrix} -12 & -22 & 8 \\ -14 & -15 & 8 \\ -36 & -66 & 20 \end{bmatrix} \end{aligned}$$

m. Fa

$$\begin{aligned} Fa &= \begin{bmatrix} -2 & -2 & 4 \\ -3 & -1 & 4 \\ -6 & -6 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \times 1 + (-2) \times 2 + 4 \times 3 \\ -3 \times 1 + (-1) \times 2 + 4 \times 3 \\ -6 \times 1 + (-6) \times 2 + 10 \times 3 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix} \end{aligned}$$

n. b'G

$$b'G = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}' \begin{bmatrix} 4 & 2 & 0 \\ 2 & 9 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 0 \\ 2 & 9 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \times 4 + 3 \times 2 + 1 \times 0 & 0 \times 2 + 3 \times 9 + 1 \times 0 & 0 \times 0 + 3 \times 0 + 1 \times 2 \end{bmatrix}$$

$$= [6 \quad 27 \quad 2]$$

o. A + 2D

$$A + 2D = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} + 2 \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 8 & 4 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 6 \\ 6 & 7 \end{bmatrix}$$

p. B'D

$$B'D = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 4 & 5 \end{bmatrix}' \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 4 + 3 \times 2 & 2 \times 2 + 3 \times 2 \\ 1 \times 4 + 4 \times 2 & 1 \times 2 + 4 \times 2 \\ 6 \times 4 + 5 \times 2 & 6 \times 2 + 5 \times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 10 \\ 12 & 10 \\ 34 & 22 \end{bmatrix}$$

q. BG

$$BG = \begin{bmatrix} 2 & 1 & 6 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 4 & 2 & 0 \\ 2 & 9 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

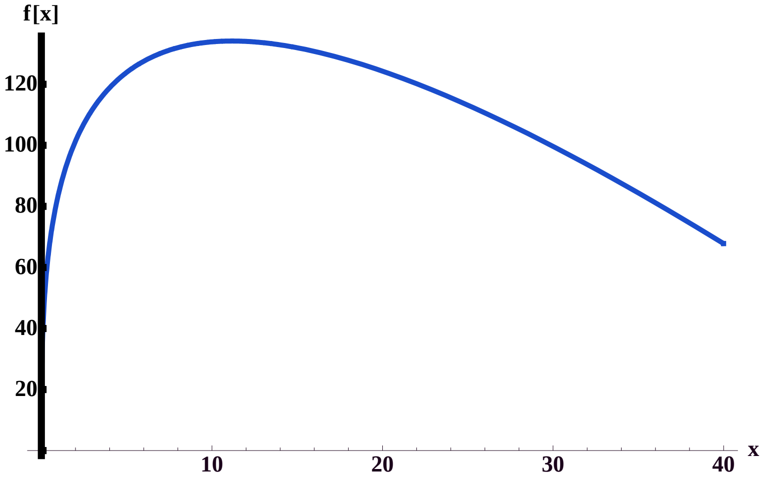
$$= \begin{bmatrix} 2 \times 4 + 1 \times 2 + 6 \times 0 & 2 \times 2 + 1 \times 9 + 6 \times 0 & 2 \times 0 + 1 \times 0 + 6 \times 2 \\ 3 \times 4 + 4 \times 2 + 5 \times 0 & 3 \times 2 + 4 \times 9 + 5 \times 0 & 3 \times 0 + 4 \times 0 + 5 \times 2 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 13 & 12 \\ 20 & 42 & 10 \end{bmatrix}$$

Problem 2. For each of the following problems, find the critical points. For each critical point state whether the function is at a relative maximum, relative minimum, or otherwise. Also find the points of inflection for each function.

a. $f(x) = 90x^{1/3} - 6x$

FIGURE 1. $f(x) = 90x^{1/3} - 6x$



The derivatives of $f(x) = 90x^{1/3} - 6x$ are given by

$$f'(x) = 30x^{-2/3} - 6 \tag{1}$$

$$f''(x) = -20x^{-5/3} \tag{2}$$

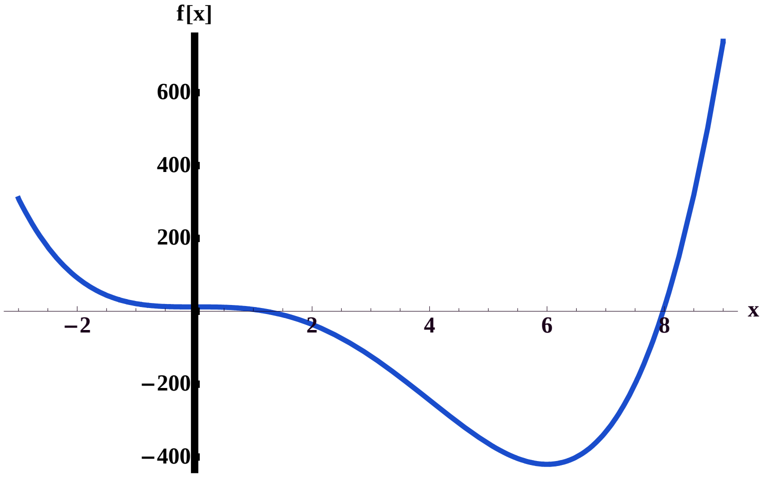
Set the first derivative, equation (1), to zero.

$$\begin{aligned} f'(x) = 30x^{-2/3} - 6 &= 0 \\ \Rightarrow \quad x^{-2/3} &= 1/5 \\ \Rightarrow \quad x &= (1/5)^{-3/2} \\ \Rightarrow \quad x &= 5\sqrt{5} \end{aligned}$$

Since the second derivative, equation (2) is smaller than zero for $x \in (0, \infty)$. So $x = 5\sqrt{5}$ is a point where the function is at a global maximum and there is no infection point of this function.

b. $f(x) = x^4 - 8x^3 + 12$

FIGURE 2. $f(x) = x^4 - 8x^3 + 12$



The derivatives of $f(x) = x^4 - 8x^3 + 12$ are given by

$$f'(x) = 4x^3 - 24x^2 \tag{3}$$

$$f''(x) = 12x^2 - 48x \tag{4}$$

$$f^{(3)}(x) = 24x - 48 \tag{5}$$

Set the first derivative, equation (3), to zero.

$$\begin{aligned} f'(x) = 4x^3 - 24x^2 &= 0 \\ \Rightarrow \quad x^2(x-6) &= 0 \\ \Rightarrow \quad x = 0 &\quad \text{or} \quad x = 6 \end{aligned}$$

When $x = 0$, the second derivative is given by

$$f''(0) = 12^2 - 48 \times 0 = 0$$

And the third derivative is given by

$$f^{(3)}(0) = 24 \times 0 - 48 < 0$$

So $x = 0$ is not a point where the function is at relative and/or global maximum/minimum. However, since when $x = 0$, the second derivative is zero and the third derivative is not zero, $x = 0$ is an inflection point of this function.

When $x = 6$, the second derivative is given by

$$f''(6) = 12 \times 6^2 - 48 \times 6 = 144 > 0$$

So $x = 6$ is a point where the function is at global minimum.

Set the second derivative, equation (4), to zero.

$$\begin{aligned} f''(x) = 12x^2 - 48x &= 0 \\ \Rightarrow \quad x(x-4) &= 0 \\ \Rightarrow \quad x = 0 &\quad \text{or} \quad x = 4 \end{aligned}$$

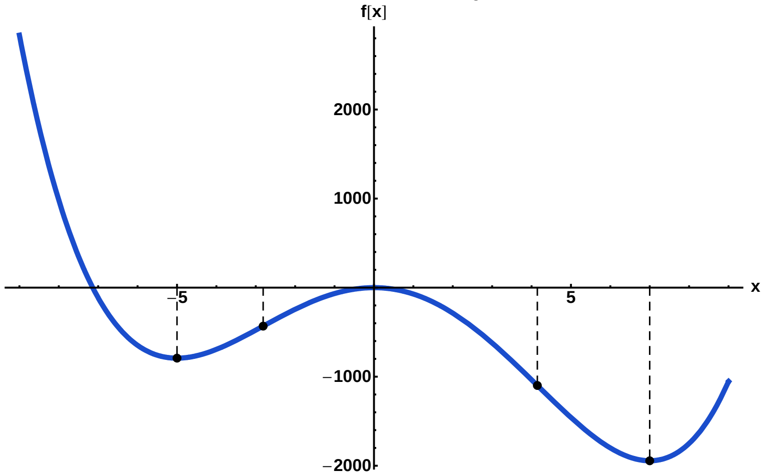
We have already know that $x = 0$ is an inflection point. And when $x = 4$, the third derivative, equation (5), is given by

$$f^{(3)}(4) = 24 \times 4 - 48 = 48 \neq 0$$

So $x = 4$ is also an inflection point.

c. $f(x) = x^4 - \frac{8}{3}x^3 - 70x^2$
Hint: The inflection points are $x = \frac{2 \pm \sqrt{109}}{3}$

FIGURE 3. $f(x) = x^4 - \frac{8}{3}x^3 - 70x^2$



The derivatives of $f(x) = x^4 - \frac{8}{3}x^3 - 70x^2$ are given by

$$f'(x) = 4x^3 - 8x^2 - 140x \tag{6}$$

$$f''(x) = 12x^2 - 16x - 140 \tag{7}$$

$$f^{(3)}(x) = 24x - 16 \tag{8}$$

Set the first derivative, equation (6), to zero.

$$\begin{aligned} f'(x) = 4x^3 - 8x^2 - 140x &= 0 \\ \Rightarrow \qquad 4x(x-7)(x+5) &= 0 \\ \Rightarrow \qquad x = 0 \quad \text{or} \quad x = 7 \quad & \text{or} \quad x = -5 \end{aligned}$$

When $x = 0,\ 7,\ -5$, the second derivatives are respectively given by

$$f''(0) = 12 \times 0^2 - 16 \times 0 - 140 = -140 < 0$$

$$f''(7) = 12 \times 7^2 - 16 \times 7 - 140 = 336 > 0$$

$$f''(-5) = 12 \times (-5)^2 - 16 \times (-5) - 140 = 240 > 0$$

So $x = 0$ is a point where the function is at a relative maximum; $x = -5$ and $x = 7$ are points where the function is at a relative minimum. Further, $f(7) < f(-5)$, hence $x = 7$ is a point where the function is at a global minimum.

Set the second derivative, equation (7), to zero.

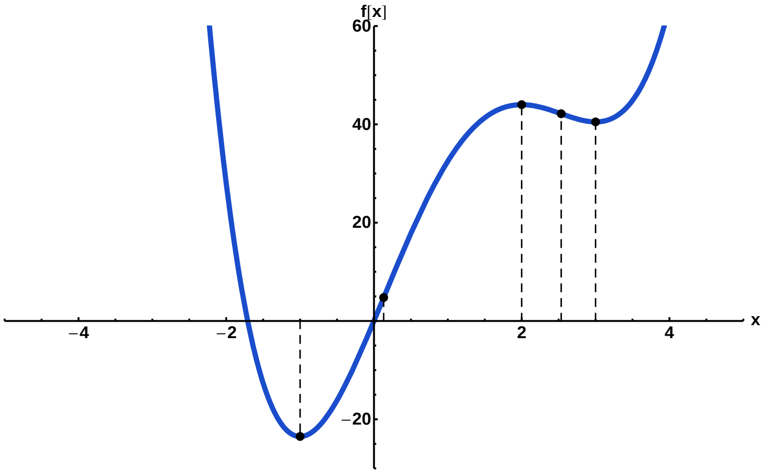
$$\begin{aligned} f''(x) = 12x^2 - 16x - 140 &= 0 \\ \Rightarrow \qquad 3x^2 - 4x - 35 &= 0 \\ \Rightarrow \qquad x = \frac{4+\sqrt{4^2-4\times3\times(-35)}}{2\times3} \quad & \text{or} \quad x = \frac{4-\sqrt{4^2-4\times3\times(-35)}}{2\times3} \\ \Rightarrow \qquad x = \frac{2+\sqrt{109}}{3} \quad & \text{or} \quad x = \frac{2-\sqrt{109}}{3} \end{aligned}$$

Since the third derivative, equation (8), is zero only when $x = 2/3$. So $x = \frac{2+\sqrt{109}}{3}$ and $x = \frac{2-\sqrt{109}}{3}$ are inflection points of this function.

d. $f(x) \;=\; \frac{3}{2}x^4 - 8x^3 + 3x^2 + 36x$

Hint: The inflection points are $x = \frac{4 \pm \sqrt{13}}{3}$

FIGURE 4. $f(x) \;=\; \frac{3}{2}x^4 - 8x^3 + 3x^2 + 36x$



Find the first and second derivatives of the function.

$$\begin{aligned} f'(x) \; &= 6x^3 - 24x^2 + 6x + 36 \\ f''(x) \; &= 18x^2 - 48x + 6 \end{aligned}$$

Set the first derivative equal to zero.

$$f'(x) \;=\; 6x^3 - 24x^2 + 6x + 36 \;=\; 0$$

If there are integer roots to this equation,we can factor the cubic if we know one of them. We can guess that x = -1 is a root of this equation so we know that we must have $(x + 1)(x + a)(x + b) = 6x^3 - 24x^2 + 6x + 36 = 0$. So if we divide $6x^3 - 24x^2 + 6x + 36$ by (x+1) we can obtain a quadratic represented by (x+a)(x+b). Hopefully we can factor this expression. So let's begin.

If we divide $6x^3 - 24x^2 + 6x + 36$ by (x+1) we first obtain

$$\begin{array}{r|l} & 6x^2 \\ \hline x+1) & 6x^3 - 24x^2 \;\; + 6x + 36 \\ & -\,6x^3 \;\; -\,6x^2 \\ \hline \end{array}$$

$$\begin{array}{r|l} & 6x^2 \\ \hline x+1) & 6x^3 - 24x^2 \;\; + 6x + 36 \\ & -\,6x^3 \;\; -\,6x^2 \\ \hline & \qquad -\,30x^2 \;\; + 6x \end{array}$$

Continuing we obtain

$$\begin{array}{r r}
 & 6x^2 - 30x \\ \cline{2-2}
x+1 \big) & 6x^3 - 24x^2 \quad + 6x + 36 \\
 & -\,6x^3 \quad -\,6x^2 \\ \cline{2-2}
 & -\,30x^2 \quad + 6x \\
 & 30x^2 + 30x \\ \cline{2-2}
\end{array}$$

and then

$$\begin{array}{r r}
 & 6x^2 - 30x \\ \cline{2-2}
x+1 \big) & 6x^3 - 24x^2 \quad + 6x + 36 \\
 & -\,6x^3 \quad -\,6x^2 \\ \cline{2-2}
 & -\,30x^2 \quad + 6x \\
 & 30x^2 + 30x \\ \cline{2-2}
 & 36x + 36
\end{array}$$

$$\begin{array}{r r}
 & 6x^2 - 30x + 36 \\ \cline{2-2}
x+1 \big) & 6x^3 - 24x^2 \quad + 6x + 36 \\
 & -\,6x^3 \quad -\,6x^2 \\ \cline{2-2}
 & -\,30x^2 \quad + 6x \\
 & 30x^2 + 30x \\ \cline{2-2}
 & 36x + 36
\end{array}$$

$$\begin{array}{r r}
 & 6x^2 - 30x + 36 \\ \cline{2-2}
x+1 \big) & 6x^3 - 24x^2 \quad + 6x + 36 \\
 & -\,6x^3 \quad -\,6x^2 \\ \cline{2-2}
 & -\,30x^2 \quad + 6x \\
 & 30x^2 + 30x \\ \cline{2-2}
 & 36x + 36 \\
 & -\,36x - 36 \\ \cline{2-2}
 & 0
\end{array}$$

We can now factor the remaining term as follows

$$\begin{aligned} 6x^2 - 30x + 36 &= 6(x^2 - 5x + 6) \\ &= 6(x-3)(x-2) \end{aligned}$$

We then have

$$\begin{aligned} (x+1)6(x-3)(x-2) &= 0 \\ \Rightarrow x = -1, x = 3, x = 2 \end{aligned}$$

We can check each of the roots in the second derivative.

$$\begin{aligned} f''(-1) &= 18(-1)^2 - 48(-1) + 6 \\ &= 18 + 48 + 6 \;=\; 72 \end{aligned}$$

So x = -1 is a minimum point.

$$\begin{aligned} f''(2) &= 18(2)^2 - 48(2) + 6 \\ &= 72 - 96 + 6 \;=\; -18 \end{aligned}$$

So x = 2 is a maximum point.

$$\begin{aligned} f''(3) &= 18(3)^2 - 48(3) + 6 \\ &= (18)(9) - (48)(3) + 6 \\ &= 162 - 144 + 6 \;=\; 24 \end{aligned}$$

So x = 3 is a minimum point.
To find the points of inflection we set the second derivative equal to zero.

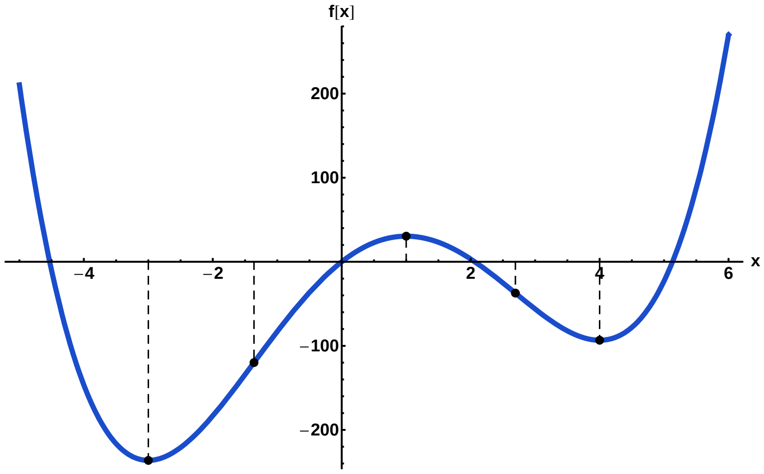
$$\begin{aligned} f''(x) &= 18(x)^2 - 48(x) + 6 \;=\; 0 \\ &= 6(3x^2 - 8x + 1) \;=\; 0 \end{aligned}$$

And unfortunately it does not factor so we obtain

$$\begin{aligned} x &= \frac{8 \pm \sqrt{64 - (4)(3)(1)}}{6} \\ &= \frac{8 \pm \sqrt{52}}{6} \\ &= \frac{8 \pm 2\sqrt{13}}{6} \\ &= \frac{4 \pm \sqrt{13}}{3} \end{aligned}$$

e. $f(x) = \frac{5}{4}x^4 - \frac{10}{3}x^3 - \frac{55}{2}x^2 + 60x$
Hint: The inflection points are $x = \frac{2\pm\sqrt{37}}{3}$

FIGURE 5. $f(x) = \frac{5}{4}x^4 - \frac{10}{3}x^3 - \frac{55}{2}x^2 + 60x$



The derivatives of $f(x) = \frac{5}{4}x^4 - \frac{10}{3}x^3 - \frac{55}{2}x^2 + 60x$ are given by

$$f'(x) = 5x^3 - 10x^2 - 55x + 60 \tag{9}$$

$$f''(x) = 15x^2 - 20x - 55 \tag{10}$$

$$f^{(3)}(x) = 30x - 20 \tag{11}$$

Set the first derivative, equation (9), to be zero.

$$f'(x) = 5x^3 - 10x^2 - 55x + 60 = 0 \tag{12}$$

By plugging in $x = 1$ into equation (12), we know that $f'(1) = 0$. So $x = 1$ is a root of equation (12) and hence we know that equation (12) can be factorized using $(x-1)$. That is, $5x^3 - 10x^2 - 55x + 60 = (x-1)(ax^2 + bx + c)$. So we try to factor $5x^3 - 10x^2 - 55x + 60$ by $(x-1)$.

$$\begin{array}{r|rrrr}
 & & 5x^2 & -5x & -60 \\ \hline
x-1 & 5x^3 & -10x^2 & -55x & +60 \\
 & -5x^3 & +5x^2 & & \\ \hline
 & & -5x^2 & -55x & \\
 & & 5x^2 & -5x & \\ \hline
 & & & -60x & +60 \\
 & & & 60x & -60 \\ \hline
 & & & & 0
\end{array}$$

Then,

$$\begin{aligned}
f'(x) = 5x^3 - 10x^2 - 55x + 60 &= 0 \\
\Rightarrow \quad (x-1)(5x^2 - 5x - 60) &= 0 \\
\Rightarrow \quad 5(x-1)(x^2 - x - 12) &= 0 \\
\Rightarrow \quad 5(x-1)(x-4)(x+3) &= 0 \\
\Rightarrow \quad x = 1 \quad \text{or} \quad x = 4 \quad &\text{or} \quad x = -3
\end{aligned}$$

When $x = 1,\ x = 4,\ x = -3$, the second derivatives are respectively given by

$$f''(1) = 15 \times 1^2 - 20 \times 1 - 55 = -60$$

$$f''(4) = 15 \times 4^2 - 20 \times 4 - 55 = 105$$

$$f''(-3) = 15 \times (-3)^2 - 20 \times (-3) - 55 = 140$$

So $x = 1$ is a relative maximum point. $x = 4$ and $x = -3$ are relative minimum points. Further, $f(4) > f(-3)$, so $x = -3$ is a global minimum point.

Set the second derivative, equation (10), to zero.

$$\begin{aligned} f''(x) = 15x^2 - 20x - 55 &= 0 \\ \Rightarrow \qquad 3x^2 - 4x - 11 &= 0 \\ \Rightarrow \qquad x &= \tfrac{4 \pm \sqrt{(-4)^2 - 4 \times 3 \times (-11)}}{2 \times 3} \\ \Rightarrow \qquad x &= \tfrac{2 \pm \sqrt{37}}{3} \end{aligned}$$

Since the third derivative, equation (11), is zero only when $x = 2/3$. So $x = \frac{2 \pm \sqrt{37}}{3}$ are inflection points.

Problem 3. Find the definite integral of each of the following functions.

a. $\int_1^3 \left(4x^3 - 3x^2 - 5x\right)\, dx$

$$\begin{aligned}\int_1^3 \left(4x^3 - 3x^2 - 5x\right)\, dx &= \left(x^4 - x^3 - 5x^2/2\right)\Big|_1^3 \\ &= 3^4 - 3^3 - 5 \times 3^2/2 - (1 - 1 - 5/2) \\ &= 34\end{aligned}$$

b. $\int_2^6 e^{4x}\, dx$

$$\begin{aligned}\int_2^6 e^{4x}\, dx &= \left.\frac{e^{4x}}{4}\right|_2^6 \\ &= \frac{e^{24} - e^8}{4}\end{aligned}$$

Problem 4. Solve the following systems of equations.

$$\begin{aligned} 81x_1^{-4/5}x_2^{2/3} - 25 &= 0 \\ 270x_1^{1/5}x_2^{-1/3} - 162 &= 0 \end{aligned}$$

From the first equation,

$$\begin{aligned} 81x_1^{-4/5}x_2^{2/3} - 25 &= 0 \\ \Rightarrow \quad 81x_1^{-4/5} &= 25x_2^{-2/3} \\ \Rightarrow \quad 9x_1^{-2/5}/5 &= x_2^{-1/3} \\ \Rightarrow \quad x_2^{-1/3} &= 9x_1^{-2/5}/5 \end{aligned}$$

Substitute $x_2^{-1/3} = 9x_1^{-2/5}/5$ into the second equation.

$$\begin{aligned} 270x_1^{1/5}x_2^{-1/3} - 162 &= 0 \\ \Rightarrow \quad 270x_1^{1/5}(9x_1^{-2/5}/5) - 162 &= 0 \\ \Rightarrow \quad x_1^{-1/5} &= \tfrac{162\times5}{270\times9} = \tfrac{1}{3} \\ \Rightarrow \quad x_1 &= 243 \end{aligned}$$

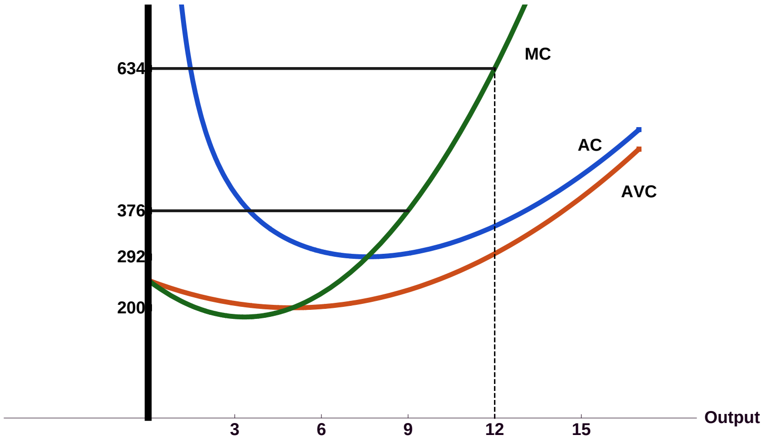
Substitute $x_1 = 243$ into $x_2^{-1/3} = 9x_1^{-2/5}/5$.

$$\begin{aligned} x_2^{-1/3} &= 9x_1^{-2/5}/5 \\ \Rightarrow \quad x_2^{-1/3} &= 9 \times 243^{-2/5}/5 = 1/5 \\ \Rightarrow \quad x_2 &= 125 \end{aligned}$$

So the solution is $x_1 = 243,\ x_2 = 125$.

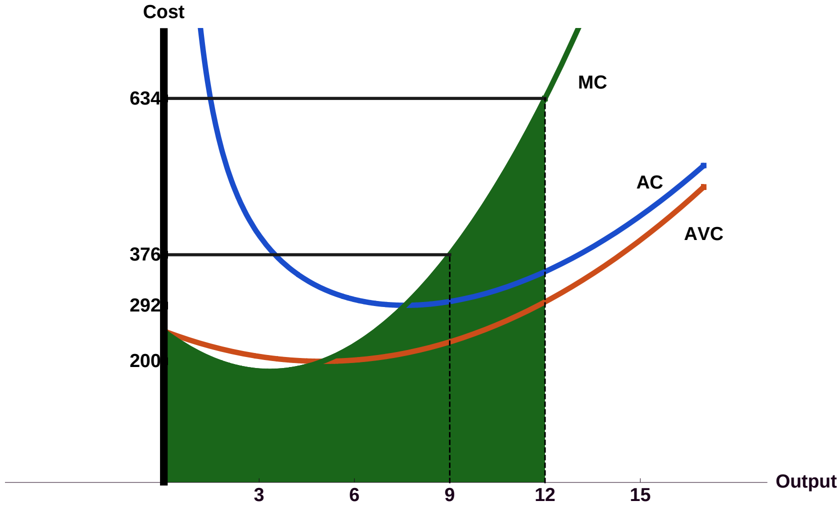
Problem 5. The cost function for a firm is a rule or mapping that tells the total cost of production of any output level produced by the firm. If the variable y represents the output of the firm, then the cost function is given by c(y). Marginal cost represents the change in the cost of production for the firm as output changes and is given by the derivative of the cost function with respect to output,i.e., Marginal Cost(MC) $= \frac{dc(y)}{dy}$. A competitive firm facing a fixed output price maximizes profit at the output level where marginal cost is equal to price as in the figure 6.

Figure 6. Profit Maximization



The area below the cost curve is a measure of variable cost and can be found by integrating the marginal cost curve from 0 to any given output level y. The shaded area in figure 7 represents the variable cost of production for the cost function $c(y) = 600 + 250y - 20\,y^2 + 2y^3$.

Figure 7. Variable Cost of Production and Producer Surplus



Producer surplus is the area below a given price and above the marginal cost curve. Producer surplus is the unshaded area below the horizontal line at 634 in figure 7. Producer surplus can be computed by subtracting the shaded area from total revenue.

a. Find the profit maximizing level of output for the following firm. Demonstrate that the level you choose maximizes profit.

$$\begin{aligned} price &= p = \$634 \\ cost &= c(y) = 600 + 250y - 20y^2 + 2y^3 \end{aligned}$$

The profit is given by

$$\begin{aligned} Profit &= Revenue - cost \\ &= 634y - (600 + 250y - 20y^2 + 2y^3) \\ &= -2y^3 + 20y^2 + 384y - 600 \end{aligned}$$

Find the first derivative of the profit with respect to y are set it to zero.

$$\begin{aligned} \frac{d\left(-2y^3+20y^2+384y-600\right)}{dy} &= 0 \\ \Rightarrow \quad -6y^2 + 40y + 384 &= 0 \\ \Rightarrow \quad -(y-12)(6y+32) &= 0 \\ \Rightarrow \quad y = 12 \quad \text{or} \quad y = -32/6 \end{aligned}$$

When $y = -32/6$, $\frac{d^2\left(-2y^3+20y^2+384y-600\right)}{dy^2} = -12y + 40 = 104 > 0$.
When $y = 12$, $\frac{d^2\left(-2y^3+20y^2+384y-600\right)}{dy^2} = -12y + 40 = -104 < 0$.
So $y = 12$ is a point of maximizing profit.

b. Find the profit maximizing level of output when the price is \$376. Demonstrate that the level you choose maximizes profit.

The profit is given by

$$\begin{aligned} Profit &= Revenue - cost \\ &= 376y - (600 + 250y - 20y^2 + 2y^3) \\ &= -2y^3 + 20y^2 + 126y - 600 \end{aligned}$$

Find the first derivative of the profit with respect to y are set it to zero.

$$\begin{aligned} \frac{d\left(-2y^3+20y^2+126y-600\right)}{dy} &= 0 \\ \Rightarrow \quad -6y^2 + 40y + 126 &= 0 \\ \Rightarrow \quad -(y-9)(6y+14) &= 0 \\ \Rightarrow \quad y = 9 \quad \text{or} \quad y = -14/6 \end{aligned}$$

When $y = -14/6$, $\frac{d^2\left(-2y^3+20y^2+126y-600\right)}{dy^2} = -12y + 40 = 68 > 0$.
When $y = 9$, $\frac{d^2\left(-2y^3+20y^2+126y-600\right)}{dy^2} = -12y + 40 = -68 < 0$.
So $y = 9$ is a point of maximizing profit when the price is \$376.

c. Explain in words why setting price equal to marginal cost and solving for the optimal output y gives the same answers as taking the derivative of profit with respect to y, setting the result equal to zero and solving for the optimal y. Remember that

$$
\begin{aligned}
Profit =& py - c(y) \\
Profit =& 634y - \left[600 \ + \ 250y \ - \ 20\ y^2 \ + \ 2y^3\right]
\end{aligned}
$$

Setting the price equal to marginal cost is to establish below equation.

$$p = MC(y) = \frac{d\,c(y)}{d\,y} \tag{13}$$

And by maximizing the profit by setting its derivative to zero gives equation as follows.

$$
\begin{aligned}
\tfrac{d\,Profit}{d\,y} = \tfrac{d\,(py - c(y))}{d\,y} \quad &= \quad 0 \\
\Leftrightarrow \qquad p - \tfrac{d\,c(y)}{dy} \quad &= \quad 0
\end{aligned} \tag{14}
$$

It is obvious that equation (13) is the same with equation (14). Therefore setting price equal to marginal cost and solving the optimal y gives the same answers as taking the derivative of profit with respect to y since these two methods will reach the same equation.

d. What is variable cost for this firm when it maximizes profit with a price of $634?

When the price is $634, the output to maximize profit is $y = 12$. The variable cost is given by

$$
\begin{aligned}
250y - 20y^2 + 2y^3 &= 250 \times 12 - 20 \times 12^2 + 2 \times 12^3 \\
&= 3576
\end{aligned}
$$

e. What is producer surplus for this profit maximizing firm when the price is $634?

The output $y = 12$ for maximizing profit when the price is $634. And the producer surplus is given by

$$
\begin{aligned}
Producer\ surplus &= 634 \times 12 - \int_0^{12} \frac{d\,\left(600 + 250y - 20y^2 + 2y^3\right)}{d\,y} d\,y \\
&= 7608 - \left(250y - 20y^2 + 2y^3\right)\Big|_0^{12} \\
&= 7608 - 3576 \\
&= 4032
\end{aligned}
$$

f. What is variable cost for this firm when it maximizes profit with a price of $376?

When the price is $376, the output to maximize profit is $y = 9$. The variable cost is given by

$$
\begin{aligned}
250y - 20y^2 + 2y^3 &= 250 \times 9 - 20 \times 9^2 + 2 \times 9^3 \\
&= 2088
\end{aligned}
$$

g. What is producer surplus for this profit maximizing firm when the price is $376?

The output $y = 9$ for maximizing profit when the price is $376. And the producer surplus is given by

$$
\begin{aligned}
Producer\ surplus &= 376 \times 9 - \left(250y - 20y^2 + 2y^3\right)\Big|_0^{9} \\
&= 3384 - 2088 \\
&= 1296
\end{aligned}
$$

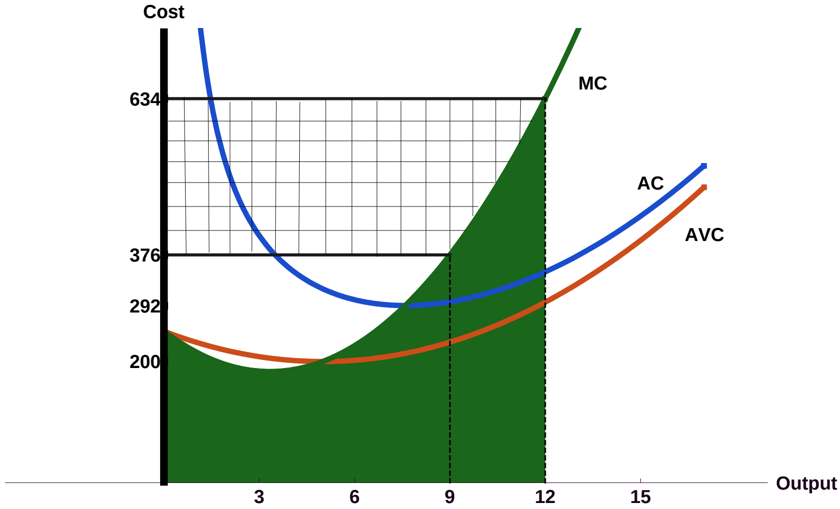
h. How much is the firm worse off when price falls from \$634 to \$376?

The decrease of producer surplus when price falls from \$634 to \$376 is given by

$$4032 - 1296 = 2736$$

i. Cross-hatch the change in producer surplus in Figure 7.

Figure 8. Change in producer surplus in Figure 7



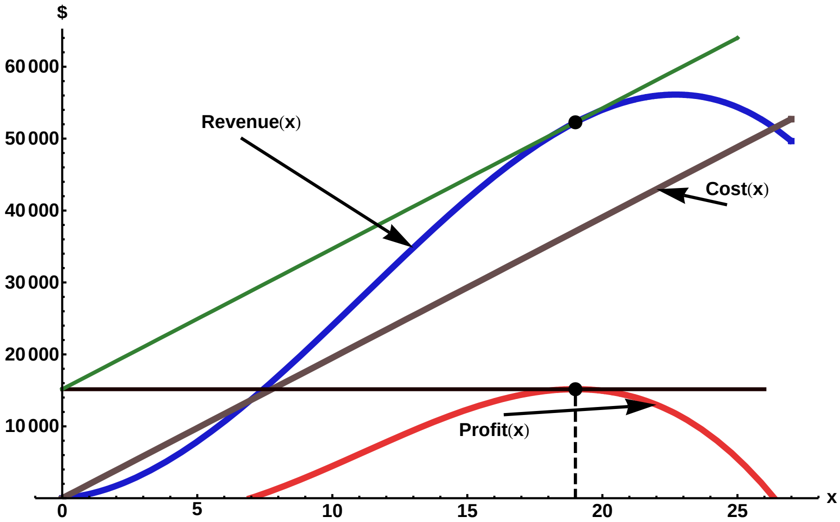
Problem 6. In the following problem you are given a production function for a firm where y is the level of output and x is the level of the variable input. You are given the price (p) of the output and the price (w) of the single variable input.

$$output\ price = p = 3$$

$$input\ price = w = 1953$$

$$y = output = f(x) = 100x + 100x^2 - 3x^3$$

Figure 9. Revenue, Cost and Profit



a. Write down an equation that represents profit for the firm.

The profit for the firm is given by

$$\begin{aligned} Profit &= py - wx \\ &= 3(100x + 100x^2 - 3x^3) - 1953x \\ &= -9x^3 + 300x^2 - 1653x \end{aligned}$$

b. Maximize this function by taking its derivative with respect to the variable input x and setting the resulting equation equal to zero.

$$\begin{aligned} \frac{d\left(-9x^3+300x^2-1653x\right)}{d\,x} &= 0 \\ \Rightarrow \quad -27x^2+600x-1653 &= 0 \\ \Rightarrow \quad 3(x-19)(29-9x) &= 0 \\ \Rightarrow \quad x = 19 \quad & \text{or} \quad x = 29/9 \end{aligned}$$

c. If you identify more than one critical value from setting the first derivative of profit equal to zero, show which ones, if any, maximize profit.

Checking the second derivatives for every critical values.

$$\frac{d^2\ \left(-9x^3+300x^2-1653x\right)}{d\,x^2} = -54x+600$$

And when $x = 19$, the second derivative $-54x+600 = -54\times 19+600 = -426 > 0$. And when $x = 29/9$, the second derivative $-54x+600 = -54\times 29/9+600 = 426 > 0$. So $x = 19$ is critical point of maximizing profit.

d. What is the optimal level of output for this firm?

For the optimal input variable $x = 19$, the corresponding optimal level of output is given by

$$\begin{aligned} f(19) &= 100 \times 19 + 100 \times 19^2 - 3 \times 19^3 \\ &= 17423 \end{aligned}$$

e. Explain in words why the slope of the total revenue function for this firm is equal to the price of the single variable input at the profit maximizing level of input use. You can use the graph 9 and the following information in explaining this phenomenon.

$$\begin{aligned} Revenue &= pf(x) \\ Cost &= wx \\ Profit &= Revenue - Cost = pf(x) - wx \end{aligned}$$

The slope of the total revenue function for this firm is given by its first derivative. That is,

$$slope = \frac{d\,Revenue}{d\,x} = \frac{d\,pf(x)}{d\,x} = pf'(x) \tag{15}$$

In order to maximize the profit, we let $\frac{d\,Profit}{d\,x}$ be zero. That is,

$$\begin{aligned} \frac{d\,Profit}{d\,x} = \frac{d\,(pf(x)-wx)}{d\,x} &= 0 \\ \Rightarrow \qquad pf'(x) - w &= 0 \end{aligned}$$

Since above equation is equivalent to $pf'(x) = w$. That is, equation (15) $= w$. In other words, above equation means that the slope of the total revenue function for this firm is equal to the price of the single variable input at the profit maximizing level of input use.

Problem 7.

You do not need to do this problem. Remove it from the packet and use for practice.

Solve the following system of equations for x_1, x_2, and x_3 using the method of elimination.

$$\{x_1 \;=\; 1,\; x_2 \;=\; -1,\; x_3 \;=\; 3\}$$

$$\begin{aligned} x_1 + 4x_2 + 3x_3 \; &= 6 \\ -2x_1 - 9x_2 - 8x_3 \; &= \; -17 \\ -3x_1 - 8x_2 + 0x_3 \; &= 5 \end{aligned}$$

Multiply the fist equation by 2 and add it to the second equation.

$$2x_1 + 8x_2 + 6x_3 + (-2x_1 - 9x_2 - 8x_3) = 12 - 17$$

That is,

$$-x_2 - 2x_3 = -5 \tag{16}$$

Multiply the fist equation by 3 and add it to the second equation.

$$3x_1 + 12x_2 + 9x_3 + (-3x_1 - 8x_2 + 0x_3) = 18 + 5$$

That is,

$$4x_2 + 9x_3 = 23 \tag{17}$$

Multiply equation (16) by 4 and add it to equation (17).

$$\begin{aligned} -4x_2 - 8x_3 + (4x_2 + 9x_3) \;&=\; -20 + 23 \\ \Rightarrow \qquad\qquad x_3 \;&=\; 3 \end{aligned}$$

Multiply equation (16) by 9/2 and add it to equation (17).

$$\begin{aligned} -\tfrac{9}{2}x_2 - 9x_3 + (4x_2 + 9x_3) \;&=\; -\tfrac{45}{2} + 23 \\ \Rightarrow \qquad\qquad -\tfrac{1}{2}x_2 \;&=\; \tfrac{1}{2} \\ \Rightarrow \qquad\qquad x_2 \;&=\; -1 \end{aligned}$$

Multiply $x_2 = -1$ by 8 and add it to the third equation.

$$\begin{aligned} 8x_2 + (-3x_1 - 8x_2 + 0x_3) \;&=\; -8 + 5 \\ \Rightarrow \qquad\qquad x_1 \;&=\; 1 \end{aligned}$$

So the solution is

$$x_1 = 1,\; x_2 = -1,\; x_3 = 3$$