ECONOMICS 207
SPRING 2008
LABORATORY EXERCISE 9
KEY

Problem 1. Consider the following matrices.

$$A = \begin{bmatrix} 1 & 4 & 3 \\ -2 & -9 & -8 \\ -3 & -8 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 64 & 24 & 5 \\ -24 & -9 & -2 \\ 11 & 4 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & -2 & 2 \\ -3 & 7 & -2 \\ 3 & -7 & 3 \end{bmatrix} \qquad G = \begin{bmatrix} 7 & -8 & -10 \\ 3 & -3 & -4 \\ 0 & 1 & 1 \end{bmatrix}$$

$$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad b = \begin{bmatrix} 6 \\ -17 \\ 5 \end{bmatrix} \qquad c = \begin{bmatrix} 9 \\ -16 \\ 19 \end{bmatrix}$$

Compute the following

a. A + B

$$A + B = \begin{bmatrix} 1 & 4 & 3 \\ -2 & -9 & -8 \\ -3 & -8 & 0 \end{bmatrix} + \begin{bmatrix} 64 & 24 & 5 \\ -24 & -9 & -2 \\ 11 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1+64 & 4+24 & 3+5 \\ -2-24 & -9+(-9) & -8+(-2) \\ -3+11 & -8+4 & 0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 65 & 28 & 8 \\ -26 & -18 & -10 \\ 8 & -4 & 1 \end{bmatrix}$$

b. AB

$$AB = \begin{bmatrix} 1 & 4 & 3 \\ -2 & -9 & -8 \\ -3 & -8 & 0 \end{bmatrix} \begin{bmatrix} 64 & 24 & 5 \\ -24 & -9 & -2 \\ 11 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 64 + 4 \times (-24) + 3 \times 11 & 1 \times 24 + 4 \times (-9) + 3 \times 4 & 1 \times 5 + 4 \times (-2) + 3 \times 1 \\ -2 \times 64 - 9 \times (-24) - 8 \times 11 & -2 \times 24 - 9 \times (-9) - 8 \times 4 & -2 \times 5 - 9 \times (-2) - 8 \times 1 \\ -3 \times 64 - 8 \times (-24) + 0 \times 11 & -3 \times 24 - 8 \times (-9) + 0 \times 4 & -3 \times 5 - 8 \times (-2) + 0 \times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Date: August 5, 2008.

c. BA

$$BA = \begin{bmatrix} 64 & 24 & 5 \\ -24 & -9 & -2 \\ 11 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 3 \\ -2 & -9 & -8 \\ -3 & -8 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 64 \times 1 + 24 \times (-2) + 5 \times (-3) & 64 \times 4 + 24 \times (-9) + 5 \times (-8) & 64 \times 3 + 24 \times (-8) + 5 \times 0 \\ -24 \times 1 - 9 \times (-2) - 2 \times (-3) & -24 \times 4 - 9 \times (-9) - 2 \times (-8) & -24 \times 3 - 9 \times (-8) - 2 \times 0 \\ 11 \times 1 + 4 \times (-2) + 1 \times (-3) & 11 \times 4 + 4 \times (-9) + 1 \times (-8) & 11 \times 3 + 4 \times (-8) + 1 \times 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

d. a'F

$$a'F = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}' \begin{bmatrix} 1 & -2 & 2 \\ -3 & 7 & -2 \\ 3 & -7 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ -3 & 7 & -2 \\ 3 & -7 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \times 1 + 2 \times (-3) + 3 \times 3 & 1 \times (-2) + 2 \times 7 + 3 \times (-7) & 1 \times 2 + 2 \times (-2) + 3 \times 3 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -9 & 7 \end{bmatrix}$$

e. a'a

$$a'a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}' \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 2 \times 2 + 3 \times 3 \end{bmatrix} = \begin{bmatrix} 14 \end{bmatrix}$$

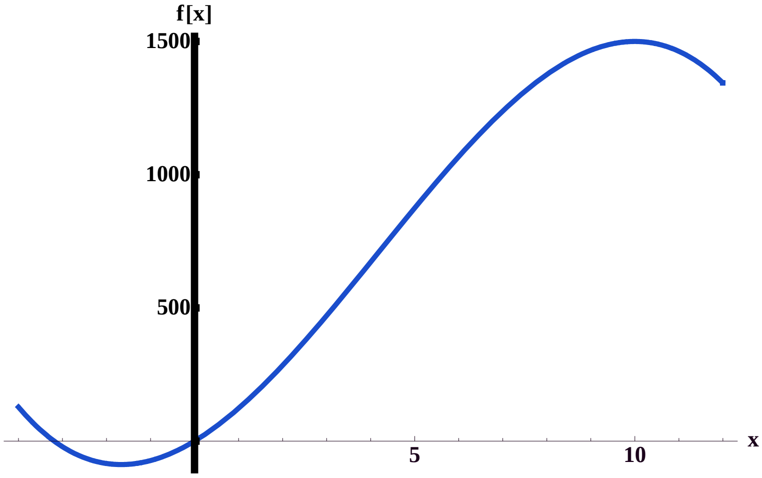
f. GF

$$GF = \begin{bmatrix} 7 & -8 & -10 \\ 3 & -3 & -4 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ -3 & 7 & -2 \\ 3 & -7 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 7 \times 1 - 8 \times (-3) - 10 \times 3 & 7 \times (-2) - 8 \times 7 - 10 \times (-7) & 7 \times 2 - 8 \times (-2) - 10 \times 3 \\ 3 \times 1 - 3 \times (-3) - 4 \times 3 & 3 \times (-2) - 3 \times 7 - 4 \times (-7) & 3 \times 2 - 3 \times (-2) - 4 \times 3 \\ 0 \times 1 + 1 \times (-3) + 1 \times 3 & 0 \times (-2) + 1 \times 7 + 1 \times (-7) & 0 \times 2 + 1 \times (-2) + 1 \times 3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 2. For each of the following problems, find the critical points. For each critical point state whether the function is at a relative maximum, relative minimum, or otherwise. Also find the points of inflection for each function.

a. $f(x) = -2x^3 + 25x^2 + 100x$
Hint: The inflection point is $x = \frac{25}{6}$

FIGURE 1. $f(x) = -2x^3 + 25x^2 + 100x$



The derivatives of $f(x) = -2x^3 + 25x^2 + 100x$ are given by

$$f'(x) = -6x^2 + 50x + 100 \tag{1}$$

$$f''(x) = -12x + 50 \tag{2}$$

$$f^{(3)}(x) = -12 \tag{3}$$

Set the first derivative, equation (1), to zero.

$$\begin{aligned} f'(x) = -6x^2 + 50x + 100 &= 0 \\ \Rightarrow \quad -(x-10)(6x+10) &= 0 \\ \Rightarrow \quad x = 10 \quad \text{or} \quad x = -10/6 \end{aligned}$$

The second derivatives for $x = 10$ and $x = -10/6$ are respectively given by

$$f''(10) = -12 \times 10 + 50 = -70$$

$$f''(-10/6) = -12 \times (-10/6) + 50 = 70$$

So $x = 10$ is a global maximum point. $x = -10/6$ is a global minimum point.

Set the second derivative, equation (2), to zero.

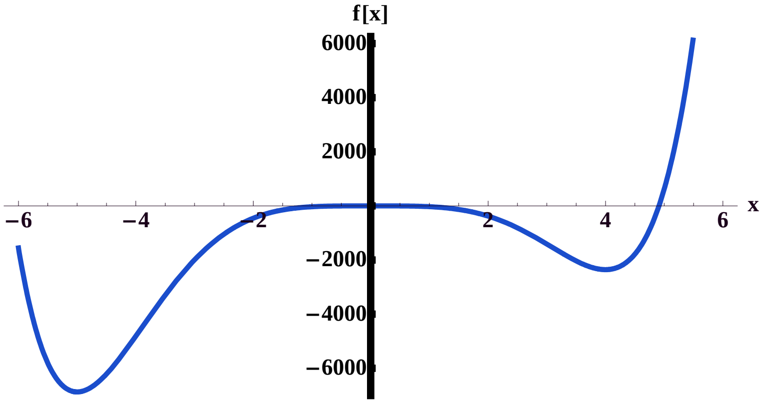
$$f''^{(x)} = -12x + 50 = 0 \quad \Rightarrow \quad x = \frac{25}{6}$$

Since the third derivative, equation (3), is not zero, $x = \frac{25}{6}$ is an inflection point of this function.

b. $f(x) = x^6 + \frac{6}{5}x^5 - 30x^4$

Hint: The inflection points are $x = \frac{-2\pm4\sqrt{19}}{5}$

FIGURE 2. $f(x) = x^6 + \frac{6}{5}x^5 - 30x^4$



The derivatives of $f(x) = x^6 + \frac{6}{5}x^5 - 30x^4$ are given by

$$f'(x) = 6x^5 + 6x^4 - 120x^3 \tag{4}$$

$$f''(x) = 30x^4 + 24x^3 - 360x^2 \tag{5}$$

$$f^{(3)}(x) = 120x^3 + 72x^2 - 720x \tag{6}$$

$$f^{(4)}(x) = 360x^2 + 144x - 720 \tag{7}$$

Set the first derivative, equation (4), to zero.

$$\begin{aligned} f'(x) = 6x^5 + 6x^4 - 120x^3 &= 0 \\ \Rightarrow \quad x^3(x^2 + x - 20) &= 0 \\ \Rightarrow \quad x^3(x+5)(x-4) &= 0 \\ \Rightarrow \quad x = 0 \quad \text{or} \quad x = -5 \quad &\text{or} \quad x = 4 \end{aligned}$$

When $x = 0$, the second derivative, equation (5), and the third derivative, equation (6), are both zero, but the fourth derivative, equation (7), is -720, which is less than 0. So $x = 0$ is a local maximum point.

When $x = -5$ and $x = 4$, the second derivative, equation (5), are respectively given by

$$f''(-5) = 30 \times (-5)^4 + 24 \times (-5)^3 - 360 \times (-5)^2 = 6750$$

$$f''(4) = 30 \times 4^4 + 24 \times 4^3 - 360 \times 4^2 = 3456$$

In addition, $f(-5) < f(4)$. So $x = 4$ is a relative minimum point. $x = -5$ is a global minimum point.

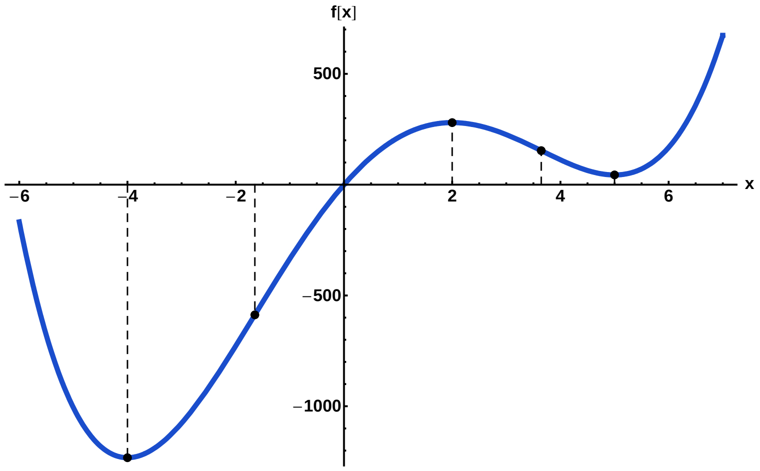
Set the second derivative, equation (5), to zero.

$$\begin{aligned} f''(x) = 30x^4 + 24x^3 - 360x^2 &= 0 \\ \Rightarrow \quad f''(x) = x^2(30x^2 + 24x - 360) &= 0 \\ \Rightarrow \quad f''(x) = x^2(5x^2 + 4x - 60) &= 0 \\ \Rightarrow \quad x = \frac{-4\pm\sqrt{4^2-4\times5\times(-60)}}{2\times5} \quad &\text{or} \quad x = 0 \\ \Rightarrow \quad x = \frac{-2\pm4\sqrt{19}}{5} \quad &\text{or} \quad x = 0 \end{aligned}$$

We have already know that $x = 0$ is a relative maximum point. When $x = \frac{-2\pm4\sqrt{19}}{5}$, the third derivative, equation (6), is not zero. So $x = \frac{-2\pm4\sqrt{19}}{5}$ are inflection points.

c. $f(x) \;=\; \frac{7}{4}x^4 - 7x^3 - 63x^2 + 280x$
Hint: The inflection points are $x = 1 \pm \sqrt{7}$

FIGURE 3. $f(x) \;=\; \frac{7}{4}x^4 - 7x^3 - 63x^2 + 280x$



The derivatives are given by

$$f'(x) = 7x^3 - 21x^2 - 126x + 280 \tag{8}$$

$$f''(x) = 21x^2 - 42x - 126 \tag{9}$$

$$f^{(3)}(x) = 42x - 42 \tag{10}$$

Set the first derivative, equation (8), to zero.

$$f'(x) = 7x^3 - 21x^2 - 126x + 280 = 0 \tag{11}$$

By guess, we know that $x = 2$ is a root of equation (11). Then we try to factorize (11) by $(x - 2)$.

$$\begin{array}{rrrrr}
 & 7x^2 & -\,7x & -\,140 & \\
\hline
x-2\,\big) & 7x^3 - 21x^2 & -\,126x & +\,280 & \\
 & -\,7x^3 + 14x^2 & & & \\
\hline
 & -\,7x^2 & -\,126x & & \\
 & 7x^2 & -\,14x & & \\
\hline
 & & -\,140x & +\,280 & \\
 & & 140x & -\,280 & \\
\hline
 & & & 0 &
\end{array}$$

Then,

$$\begin{aligned}
f'(x) = 7x^3 - 21x^2 - 126x + 280 &= 0 \\
\Rightarrow \quad (x-2)(7x^2 - 7x - 140) &= 0 \\
\Rightarrow \quad (x-2)(x^2 - x - 20) &= 0 \\
\Rightarrow \quad (x-2)(x-5)(x+4) &= 0 \\
\Rightarrow \quad x = 2 \quad \text{or} \quad x = 5 \quad &\text{or} \quad x = -4
\end{aligned}$$

When $x = 2,\ 5,\ -4$, the second derivatives, equation (9), are respectively given by

$$f''(2) = 21 \times 2^2 - 42 \times 2 - 126 = -126 < 0$$

$$f''(5) = 21 \times 5^2 - 42 \times 5 - 126 = 189 > 0$$

$$f''(-4) = 21 \times (-4)^2 - 42 \times (-4) - 126 = 378 > 0$$

In addition, $f(-4) < f(5)$. So $x = 2$ is a relative maximum point. $x = 5$ is a relative minimum point. $x = -4$ is a global minimum point.

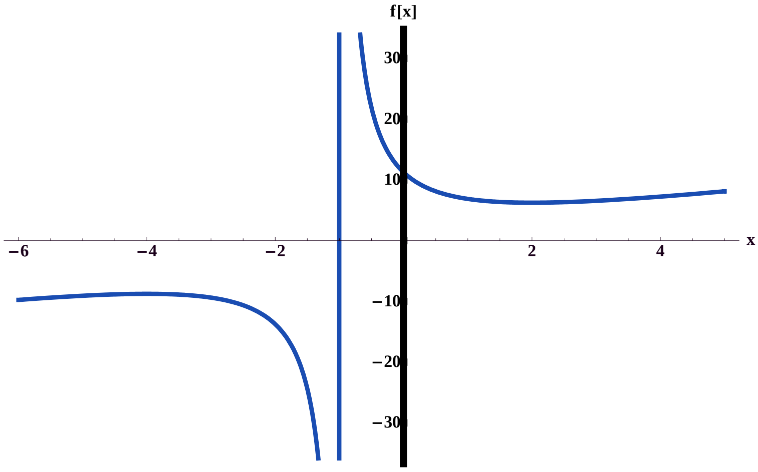
Set the second derivative, equation (9), to zero.

$$\begin{aligned} f''(x) = 21x^2 - 42x - 126 &= 0 \\ \Rightarrow \qquad x^2 - 2x - 6 &= 0 \\ \Rightarrow \qquad x = \tfrac{2 \pm \sqrt{2^2 - 4 \times 1 \times (-6)}}{2} & \\ \Rightarrow \qquad x = 1 \pm \sqrt{7} & \end{aligned}$$

Since the third derivative, equation (10), is zero only when $x = 1$, $x = 1 \pm \sqrt{7}$ are inflection pints of this function.

d. $f(x) = \frac{5x^2+5x+45}{4x+4} = \frac{5x^2+5x+45}{4(x+1)}$
Hint: The second derivative of this function is $\frac{45}{2(1+x)^3}$.

FIGURE 4. $f(x) = \frac{5x^2+5x+45}{4x+4}$



To begin with, simplify the function.

$$f(x) = \frac{5x^2+5x+45}{4x+4} = \frac{5x^2+5x+45}{4(x+1)}$$
$$= \frac{5x(x+1)+45}{4(x+1)} = \frac{5}{4}x + \frac{45}{4(x+1)}$$

Then the derivatives of above function are given by

$$f'(x) = \frac{5}{4} - \frac{45}{4}(x+1)^{-2} \tag{12}$$
$$f''(x) = \frac{45}{4} \times 2(x+1)^{-3} = \frac{45}{2}(x+1)^{-3} \tag{13}$$

Set the first derivative, equation (12), to zero.

$$\begin{aligned} f'(x) = \tfrac{5}{4} - \tfrac{45}{4}(x+1)^{-2} &= 0 \\ \Rightarrow \quad (x+1)^{-2} &= 1/9 \\ \Rightarrow \quad (x+1)^2 &= 9 \\ \Rightarrow \quad x = 2 &\text{ or } x = -4 \end{aligned}$$

When $x = 2$ and $x = -4$, the second derivative, equation (13), are respectively given by

$$f''(2) = \frac{45}{2}(2+1)^{-3} > 0$$
$$f''(-4) = \frac{45}{2}(-4+1)^{-3} < 0$$

So $x = 2$ is a local minimum point; $x = -4$ is a local maximum point.

Since the second derivative, equation (13), does not equal to zero, there is no inflection point for this function.

Problem 3. a. Use elementary row operations to solve the following system of equations. The answers are $x_1 = 1$, $x_2 = 2$

$$Qx = r$$

$$\begin{pmatrix} 1 & 4 \\ -2 & -9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 9 \\ -20 \end{pmatrix}$$

First create the augmented matrix $\tilde{A} = (Q \quad r)$.

$$\tilde{A} = (Q \quad r) = \begin{pmatrix} 1 & 4 & 9 \\ -2 & -9 & -20 \end{pmatrix}$$

Then using elementary row operations on matrix $\tilde{A}$ to create an identity matrix on the left side of the augmented matrix where the diagonal from left to right are all ones and all other entries are zeros.

Add the first row multiplied by 2 to the second row.

$$\begin{pmatrix} 1 & 4 & 9 \\ -2+1\times 2 & -9+4\times 2 & -20+9\times 2 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 9 \\ 0 & -1 & -2 \end{pmatrix} \tag{14}$$

Based on the matrix on the right side of equation (14), add the second row multiplied by 4 to the first row.

$$\begin{pmatrix} 1 & 4+(-1)\times 4 & 9+(-2)\times 4 \\ 0 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \end{pmatrix} \tag{15}$$

Based on the matrix on the right side of equation (15), multiply the second row by -1.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1\times(-1) & -2\times(-1) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \tag{16}$$

By observing above matrix, the solution for the equation is $x_1 = 1, \ x_2 = 2$.

b. Use elementary row operations to solve the following system of equations. The answers are $x_1 = 1$, $x_2 =$ -1, $x_3 = 3$.

$$Ax = b$$

$$\begin{pmatrix} 1 & 4 & 3 \\ -2 & -9 & -8 \\ -3 & -8 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -17 \\ 5 \end{pmatrix}$$

First create the augmented matrix $\tilde{A} = (A \quad b)$.

$$\tilde{A} = (A \quad b) = \begin{pmatrix} 1 & 4 & 3 & 6 \\ -2 & -9 & -8 & -17 \\ -3 & -8 & 0 & 5 \end{pmatrix}$$

Then using elementary row operations on matrix $\tilde{A}$ to create an identity matrix on the left side of the augmented matrix where the diagonal from left to right are all ones and all other entries are zeros.

To begin with, based on $\tilde{A}$, add the first row multiplied by 2 to the second row.

$$\begin{pmatrix} 1 & 4 & 3 & 6 \\ -2+1\times 2 & -9+4\times 2 & -8+3\times 2 & -17+6\times 2 \\ -3 & -8 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 3 & 6 \\ 0 & -1 & -2 & -5 \\ -3 & -8 & 0 & 5 \end{pmatrix} \tag{17}$$

Based on the matrix on the right side of (17), add the first row multiplied 3 to the third row. and add it to the second row, and multiply the first row by 3 and add it to the third row.

$$\begin{pmatrix} 1 & 4 & 3 & 6 \\ 0 & -1 & -2 & -5 \\ -3+1\times 3 & -8+4\times 3 & 0+3\times 3 & 5+6\times 3 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 3 & 6 \\ 0 & -1 & -2 & -5 \\ 0 & 4 & 9 & 23 \end{pmatrix} \tag{18}$$

Based on the matrix on the right side of (18), subtract the third row from the first row.

$$\begin{pmatrix} 1 & 4-4 & 3-9 & 6-23 \\ 0 & -1 & -2 & -5 \\ 0 & 4 & 9 & 23 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -6 & -17 \\ 0 & -1 & -2 & -5 \\ 0 & 4 & 9 & 23 \end{pmatrix} \tag{19}$$

Based on the matrix on the right side of (19), add the second row multiplied by 4 to the third row.

$$\begin{pmatrix} 1 & 0 & -6 & -17 \\ 0 & -1 & -2 & -5 \\ 0 & 4+(-1)\times 4 & 9+(-2)\times 4 & 23+(-5)\times 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -6 & -17 \\ 0 & -1 & -2 & -5 \\ 0 & 0 & 1 & 3 \end{pmatrix} \tag{20}$$

Based on the matrix on the right side of (20), add the third row multiplied by 2 to the second row.

$$\begin{pmatrix} 1 & 0 & -6 & -17 \\ 0 & -1 & -2+1\times 2 & -5+3\times 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -6 & -17 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix} \tag{21}$$

Based on the matrix on the right side of (21), add the third row multiplied by 6 to the first row.

$$\begin{pmatrix} 1 & 0 & -6+6 & -17+3\times 6 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix} \tag{22}$$

Based on the matrix on the right side of (22), multiply the second row by −1.

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 \times (-1) & 0 & 1 \times (-1) \\ 0 & 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix} \tag{23}$$

Finally, by observing the matrix on the right side of (23), the solution is $x_1 = 1,\ x_2 = -1,\ x_3 = 3$.

c. Use elementary row operations to solve the following system of equations. The answers are $x_1 = 1$, $x_2 =$ -1, $x_3 = 3$.

$$Fx = c$$

$$\begin{pmatrix} 1 & -2 & 2 \\ -3 & 7 & -2 \\ 3 & -7 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 9 \\ -16 \\ 19 \end{pmatrix}$$

First create the augmented matrix $\tilde{A} = (F \quad c)$.

$$\tilde{A} = (F \quad c) = \begin{pmatrix} 1 & -2 & 2 & 9 \\ -3 & 7 & -2 & -16 \\ 3 & -7 & 3 & 19 \end{pmatrix}$$

Then using elementary row operations on matrix $\tilde{A}$ to create an identity matrix on the left side of the augmented matrix where the diagonal from left to right are all ones and all other entries are zeros.

To begin with, based on $\tilde{A}$, add the first row multiplied by 3 to the second row.

$$\begin{pmatrix} 1 & -2 & 2 & 9 \\ -3+1\times 3 & 7+(-2)\times 3 & -2+2\times 3 & -16+9\times 3 \\ 3 & -7 & 3 & 19 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 2 & 9 \\ 0 & 1 & 4 & 11 \\ 3 & -7 & 3 & 19 \end{pmatrix} \tag{24}$$

Base on the matrix on the right side of equation (24), add the first row multiplied by -3 to the third row.

$$\begin{pmatrix} 1 & -2 & 2 & 9 \\ 0 & 1 & 4 & 11 \\ 3+1\times(-3) & -7+(-2)\times(-3) & 3+2\times(-3) & 19+9\times(-3) \end{pmatrix} = \begin{pmatrix} 1 & -2 & 2 & 9 \\ 0 & 1 & 4 & 11 \\ 0 & -1 & -3 & -8 \end{pmatrix} \tag{25}$$

Based on the matrix on the right side of (25), add the second row to the third row.

$$\begin{pmatrix} 1 & -2 & 2 & 9 \\ 0 & 1 & 4 & 11 \\ 0 & -1+1 & -3+4 & -8+11 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 2 & 9 \\ 0 & 1 & 4 & 11 \\ 0 & 0 & 1 & 3 \end{pmatrix} \tag{26}$$

Based on the matrix on the right side of (26), add the second row multiplied by 2 to the first row.

$$\begin{pmatrix} 1 & -2+2 & 2+4\times 2 & 9+11\times 2 \\ 0 & 1 & 4 & 11 \\ 0 & 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 10 & 31 \\ 0 & 1 & 4 & 11 \\ 0 & 0 & 1 & 3 \end{pmatrix} \tag{27}$$

Based on the matrix on the right side of (27), add the third row multiplied by -4 to the second row.

$$\begin{pmatrix} 1 & 0 & 10 & 31 \\ 0 & 1 & 4+1\times(-4) & 11+3\times(-4) \\ 0 & 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 10 & 31 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix} \tag{28}$$

Based on the matrix on the right side of (28), add the third row multiplied by -10 to the first row.

$$\begin{pmatrix} 1 & 0 & 10+1\times(-10) & 31+3\times(-10) \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix} \tag{29}$$

Finally, by observing the matrix on the right side of equation (29), the solution is $x_1 = 1,\ x_2 = -1,\ x_3 = 3$.

$$B = \begin{bmatrix} 64 & 24 & 5 \\ -24 & -9 & -2 \\ 11 & 4 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} 7 & -8 & -10 \\ 3 & -3 & -4 \\ 0 & 1 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 6 \\ -17 \\ 5 \end{bmatrix} \qquad c = \begin{bmatrix} 9 \\ -16 \\ 19 \end{bmatrix}$$

What is the following product?

Bb

$$Bb = \begin{bmatrix} 64 & 24 & 5 \\ -24 & -9 & -2 \\ 11 & 4 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ -17 \\ 5 \end{bmatrix} = \begin{bmatrix} 64 \times 6 + 24 \times (-17) + 5 \times 5 \\ -24 \times 6 + (-9) \times (-17) + (-2) \times 5 \\ 11 \times 6 + 4 \times (-17) + 1 \times 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

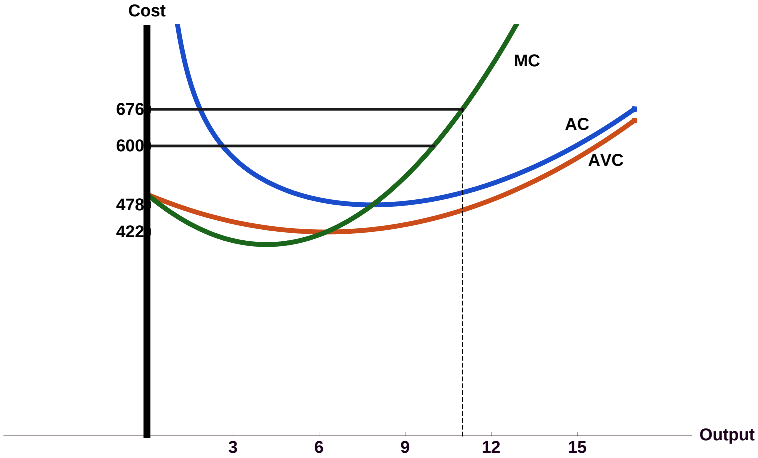
What is the following product?

Gc

$$Gc = \begin{bmatrix} 7 & -8 & -10 \\ 3 & -3 & -4 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ -16 \\ 19 \end{bmatrix} = \begin{bmatrix} 7 \times 9 + (-8) \times (-16) + (-10) \times 19 \\ 3 \times 9 + (-3) \times (-16) + (-4) \times 19 \\ 0 \times 9 + 1 \times (-16) + 1 \times 19 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

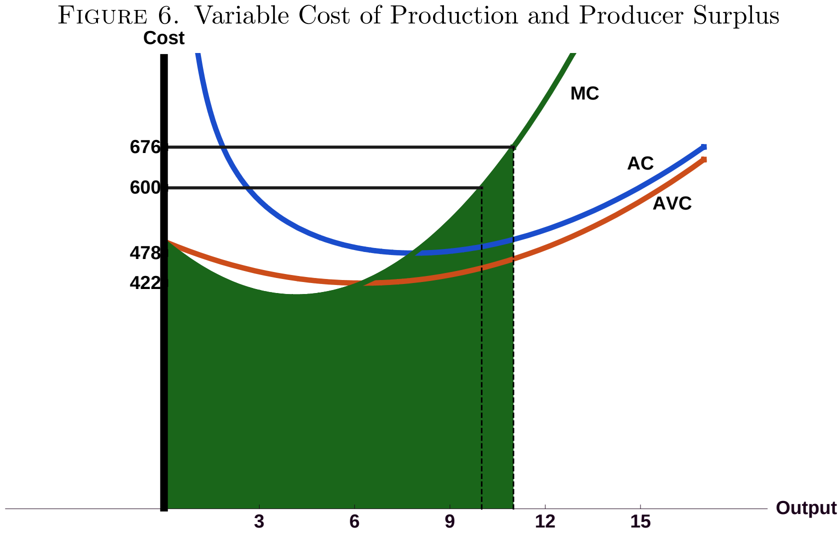
Problem 4. The cost function for a firm is a rule or mapping that tells the total cost of production of any output level produced by the firm. If the variable y represents the output of the firm, then the cost function is given by c(y). Marginal cost represents the change in the cost of production for the firm as output changes and is given by the derivative of the cost function with respect to output,i.e., Marginal Cost(MC) $= \frac{dc(y)}{dy}$. A competitive firm facing a fixed output price maximizes profit at the output level where marginal cost is equal to price as in the figure 5.

FIGURE 5. Profit Maximization



The area below the cost curve is a measure of variable cost and can be found by integrating the marginal cost curve from 0 to any given output level y. The shaded area in figure 6 represents the variable cost of production for the cost function $c(y) = 400 + 500y - 25y^2 + 2y^3$ because the integral of marginal cost is variable cost.

FIGURE 6. Variable Cost of Production and Producer Surplus



Producer surplus is the area below a given price and above the marginal cost curve. Producer surplus is the unshaded area below the horizontal line at 676 in figure 6. Producer surplus can be computed by subtracting the shaded area from total revenue.

a. Find the profit maximizing level of output for the following firm. Demonstrate that the level you choose maximizes profit.

$$\begin{aligned} price &= p &&= \$676 \\ cost &= c(y) &&= 400 + 500y - 25y^2 + 2y^3 \end{aligned}$$

When the price is \$676, the profit for the firm is given by

$$\begin{aligned} Profit &= Revenue - cost = py - c(y) \\ &= 676y - (400 + 500y - 25y^2 + 2y^3) \\ &= -2y^3 + 25y^2 + 176y - 400 \end{aligned}$$

To find the profit maximizing level of output, set the first derivative of the profit to zero.

$$\begin{aligned} \frac{d\,Profit}{d\,y} &= 0 \\ \Rightarrow \quad -6y^2 + 50y + 176 &= 0 \\ \Rightarrow \quad (y - 11)(-6y - 16) &= 0 \\ \Rightarrow \quad y = 11 &\text{ or } -\tfrac{8}{3} \end{aligned}$$

When $y = -\frac{8}{3}$, $\frac{d^2\,Profit}{d\,y^2} = -12y + 50 = -12 \times \left(-\frac{8}{3}\right) + 50 = 82 > 0$.
When $y = 11$, $\frac{d^2\,Profit}{d\,y^2} = -12y + 50 = -12 \times 11 + 50 = -82 < 0$.
So the profit maximizing level of output is $y = 11$.

b. Find the profit maximizing level of output when the price is \$600. Demonstrate that the level you choose maximizes profit.

When the price is \$600, the profit for the firm is given by

$$\begin{aligned} Profit &= Revenue - cost = py - c(y) \\ &= 600y - (400 + 500y - 25y^2 + 2y^3) \\ &= -2y^3 + 25y^2 + 100y - 400 \end{aligned}$$

To find the profit maximizing level of output, set the first derivative of the profit to zero.

$$\begin{aligned} \frac{d\,Profit}{d\,y} &= 0 \\ \Rightarrow \quad -6y^2 + 50y + 100 &= 0 \\ \Rightarrow \quad (y - 10)(-6y - 10) &= 0 \\ \Rightarrow \quad y = 10 &\text{ or } -\tfrac{5}{3} \end{aligned}$$

When $y = -\frac{5}{3}$, $\frac{d^2\,Profit}{d\,y^2} = -12y + 50 = -12 \times \left(-\frac{5}{3}\right) + 50 = 70 > 0$.
When $y = 10$, $\frac{d^2\,Profit}{d\,y^2} = -12y + 50 = -12 \times 10 + 50 = -70 < 0$.
So the profit maximizing level of output is $y = 10$.

c. Explain in words why setting price equal to marginal cost and solving for the optimal output y gives the same answers as taking the derivative of profit with respect to y, setting the result equal to zero and solving for the optimal y. Remember that

$$Profit = py - c(y)$$
$$Profit = \pi = 676y - \left[400 + 500y - 25\ y^2 + 2y^3\right]$$
$$\frac{d\pi}{dy} =$$

Setting price equal to marginal cost is given by

$$p = MC(y) = \frac{d\,c(y)}{d\,y}, \tag{30}$$

which is equivalent to

$$p - \frac{d\,c(y)}{d\,y} = 0. \tag{31}$$

The other method of taking the derivative of profit with respect to y and then setting the result to zero gives

$$\frac{d\,Profit}{d\,y} = \frac{d\,(py - c(y))}{d\,y} = p - \frac{d\,c(y)}{d\,y} = 0 \tag{32}$$

Obviously, equation (31) is equivalent to equation (32). That is, setting price equal to marginal cost and solving for the optimal output y gives the same answers as taking the derivative of profit with respect to y, setting the result equal to zero and solving for the optimal y.

d. Show that variable cost for this firm when it maximizes profit with a price of $676 is $5137.

The variable cost for this firm is given by

$$500y - 25y^2 + 2y^3$$

From part a, the output level is $y = 11$ to maximize profit with a price of $676. So the variable cost is

$$500y - 25y^2 + 2y^3 = 500 \times 11 - 25 \times 11^2 + 2 \times 11^3 = 5137$$

e. Show that producer surplus for this profit maximizing firm when the price is $676 is $2299.

The marginal cost function for this firm is given by

$$MC(y) = \frac{d\,c(y)}{d\,y} = 500 - 50y^2 + 6y^2$$

Also we know from part a that the output level $y = 11$ to maximize profit when the price is $676. The producer surplus for the firm is then given by

$$\begin{aligned} Producer\ surplus &= py - \int_0^{11} (500 - 50y^2 + 6y^2)dy \\ &= 676 \times 11 - \left(500y - 25y^2 + 2y^3\right)\Big|_0^{11} \\ &= 7436 - 5137 \\ &= 2299 \end{aligned}$$

f. Show that variable cost for this firm when it maximizes profit with a price of \$600 is \$4500.

From part b, the output level is $y = 10$ for the firm to maximize profit with a price of \$600. Then, the variable cost for this firm is given by

$$500y - 25y^2 + 2y^3 = 500 \times 10 - 25 \times 10^2 + 2 \times 10^3 = 4500$$

g. Show that producer surplus for this profit maximizing firm when the price is \$600 is \$1500.

Also we know from part b that the output level $y = 10$ to maximize profit when the price is \$600. The producer surplus for the firm is then given by

$$\begin{aligned} Producer\ surplus &= py - \int_0^{10} (500 - 50y^2 + 6y^2)dy \\ &= 600 \times 10 - \left(500y - 25y^2 + 2y^3\right)\Big|_0^{10} \\ &= 6000 - 4500 \\ &= 1500 \end{aligned}$$

h. How much is the firm worse off when price falls from \$676 to \$600?

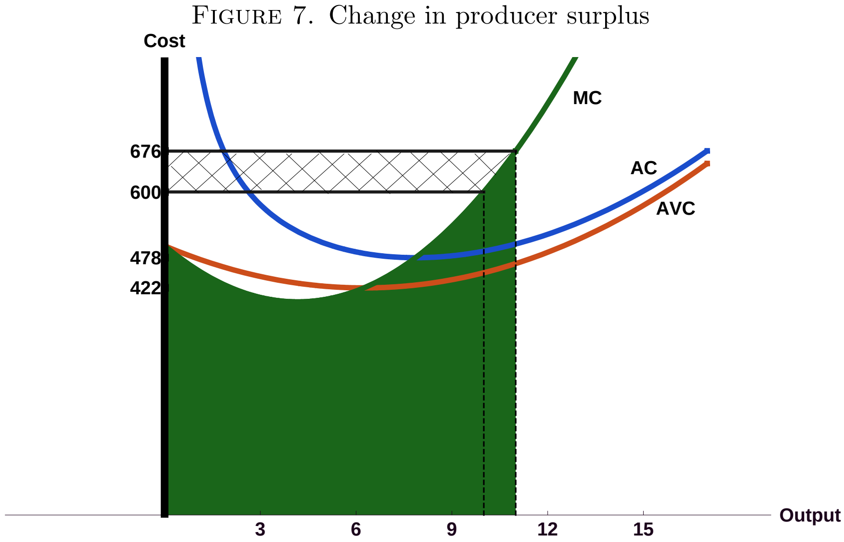
When price falls from \$676 to \$600, the producer surplus falls from \$2299 to \$1500. And

$$2299 - 1500 = 799$$

So the firm is worse off \$799 when price falls from \$676 to \$600.

i. Cross-hatch the change in producer surplus in Figure 6.

FIGURE 7. Change in producer surplus



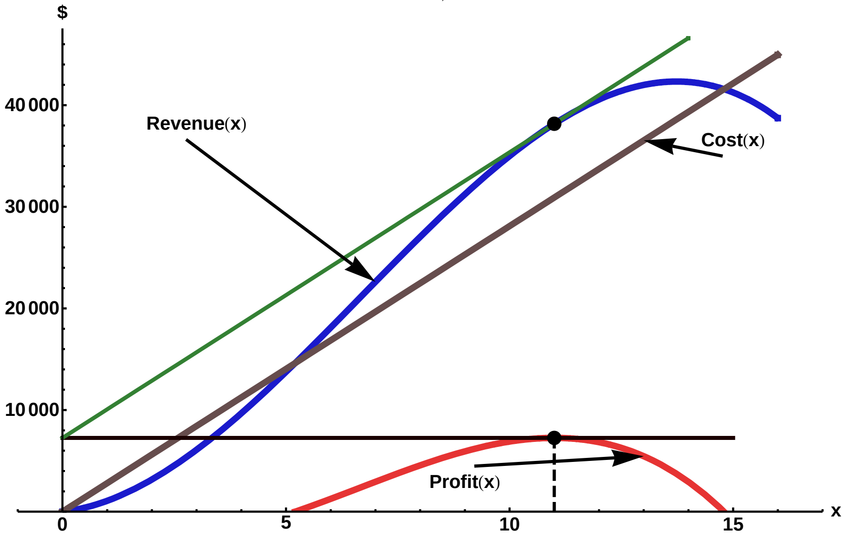
Problem 5. In the following problem you are given a production function for a firm where y is the level of output and x is the level of the variable input. You are given the price (p) of the output and the price (w) of the single variable input.

$$output\ price = p = 10$$

$$input\ price = w = 2810$$

$$y \ = \ output \ = \ f(x) \ = \ 50x + 60x^2 - 3x^3$$

FIGURE 8. Revenue, Cost and Profit



a. Write down an equation that represents profit for the firm.

The profit for the firm is given by

$$\begin{aligned} Profit &= Reveune - Cost \\ &= py - wx \\ &= 10(50x + 60x^2 - 3x^3) - 2810x \\ &= -30x^3 + 600x^2 - 2310x \end{aligned}$$

b. Maximize this function by taking its derivative with respect to the variable input x and setting the resulting equation equal to zero.

Take the derivative with respect to x and set it to zero.

$$\begin{aligned} & \frac{d\,Profit}{d\,x} &=& \ 0 \\ \Rightarrow & \frac{d\left(-30x^3+600x^2-2310x\right)}{d\,x} &=& \ 0 \\ \Rightarrow & -90x^2+1200x-2310 &=& \ 0 \\ \Rightarrow & 3x^2-40x+77 &=& \ 0 \\ \Rightarrow & (x-11)(3x-7) &=& \ 0 \\ \Rightarrow & x=11 & \text{or} & \quad x=7/3 \end{aligned}$$

c. If you identify more than one critical value from setting the first derivative of profit equal to zero, show which ones, if any, maximize profit.

The second derivative is given by

$$\begin{aligned} \frac{d^2\,Profit}{d\,x} &= \frac{d^2\ \left(-30x^3+600x^2-2310x\right)}{d\,x^2} \\ &= \frac{d\ \left(-90x^2+1200x-2310\right)}{d\,x} \\ &= -180x+1200 \end{aligned}$$

Check the second derivative when $x = 11$.

$$-180x+1200 = -180\times 11+1200 = -780$$

Check the second derivative when $x = 7/3$.

$$-180x+1200 = -180\times 7/3+1200 = 780$$

So when the input $x = 11$, the profit attains its maximum.

d. Rewrite the optimal level of input for this firm?

The optimal level of input for this firm is given by

$$x = 11$$

e. What is the optimal level of output for this firm?

The optimal level of output for this firm is given by

$$\begin{aligned} y &= f(11) = 50 \times 11 + 60 \times 11^2 - 3 \times 11^3 \\ &= 3817 \end{aligned}$$

f. Explain in words why the value of the marginal product for this firm is equal to the price of the single variable input at the profit maximizing level of input use. You can use the following information in explaining this phenomenon. Say something about the benefits of using an input not being less than the cost of the input.

$$\begin{aligned} Output =& y = f(x) \\ MP =& \text{Marginal Product} = \frac{d\,f(x)}{dx} = f'(x) = \frac{\Delta y}{\Delta x} \\ Revenue =& pf(x) \\ Cost =& wx \\ Profit = \pi =& Revenue - Cost = pf(x) - wx \\ \frac{d\pi}{dx} =& \end{aligned}$$

Setting the derivative of profit to zero and solving for the input level of maximizing profit can be described by below equation.

$$\frac{d\,Profit}{d\,x} = \frac{d\,pf(x) - wx}{d\,x} = pf'(x) - w = 0,$$

which is equivalent to

$$pf'(x) = w \tag{33}$$

The left side of equation (33) is $pf'(x)$, which is the value of marginal product as well. The right side of equation (33) is w, the price of the single variable input. So the value of the marginal product for this firm is equal to the price of the single variable input at the profit maximizing level of input use.

In other words, the value of marginal product is the marginal revenue of increasing a unit of the single variable input. So if the value of marginal product is higher than the price of the single variable input, the profit will still increase as the input increases. On the other hand, if the value of marginal product is less than the price of the single variable input, the profit will decrease as the input increases. As a result, when the value of marginal product equals the price of the single input variable, the profit attains its maximum.