For this laboratory exercise, consider the following matrices and vectors. You may want to tear this page off so it is easy to view.

\[
A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} -5 & 2 & 0 \\ 1 & -1 & -1 \\ -4 & 2 & 1 \end{bmatrix}, \\
D = \begin{bmatrix} 1 & -4 & 2 \\ -3 & 13 & -7 \\ 4 & -4 & -5 \end{bmatrix}, \quad E = \begin{bmatrix} 5 & 9 \\ 2 & 3 \end{bmatrix}, \quad F = \begin{bmatrix} 3 & 6 & -3 \\ -1 & -5/3 & 2 \\ -3 & -4 & 8 \end{bmatrix}, \\
a = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad c = \begin{bmatrix} 3 \\ -6 \\ 7 \end{bmatrix}, \quad d = \begin{bmatrix} 2 \\ -7 \\ -7 \end{bmatrix}, \quad e = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad f = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}
\]
Problem 1.
a. Use elementary row operations to solve the following system of equations.
\[ Ax = a \]
\[
\begin{pmatrix}
1 & 1 \\
2 & 3
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
1
\end{pmatrix}
\]

Create the augmented matrix \( \tilde{A} \). That is,
\[
\tilde{A} = (A \ a) = 
\begin{pmatrix}
1 & 1 & 1 \\
2 & 3 & 1
\end{pmatrix}
\]

Then use elementary row operations on matrix \( \tilde{A} \) to create an identity matrix on the left side. To begin with, based on \( \tilde{A} \), add the first row multiplied by \(-2\) to the second row.

\[
\begin{pmatrix}
1 & 1 & 1 \\
2 + 1 \times (-2) & 3 + 1 \times (-2) & 1 + 1 \times (-2)
\end{pmatrix}
= 
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & -1
\end{pmatrix}
\quad (1)
\]

Based on the matrix on the right side of equation (1), subtract the second row from the first row.

\[
\begin{pmatrix}
1 & 1 - 1 & 1 - (-1) \\
0 & 1 & -1
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & -1
\end{pmatrix}
\quad (2)
\]

So the solutions are \( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \).
b. Use elementary row operations to solve the following system of equations.

\[
\begin{pmatrix}
4 & 3 \\
2 & 2
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= 
\begin{pmatrix}
3 \\
-1
\end{pmatrix}
\]

Create the augmented matrix \( \tilde{A} \). That is,

\[
\tilde{A} = \begin{pmatrix}
4 & 3 & 3 \\
2 & 2 & -1
\end{pmatrix}
\]

Then use elementary row operations on matrix \( \tilde{A} \) to create an identity matrix on the left side. To begin with, based on \( \tilde{A} \), add the first row multiplied by \(-1/2\) to the second row.

\[
\begin{pmatrix}
4 & 3 & 3 \\
2 + 4 \times (-1/2) & 2 + 3 \times (-1/2) & -1 + 3 \times (-1/2)
\end{pmatrix}
= 
\begin{pmatrix}
4 & 3 & 3 \\
0 & 1/2 & -5/2
\end{pmatrix} \tag{3}
\]

Based on the matrix on the right side of equation (3), multiply the second row by 2.

\[
\begin{pmatrix}
4 & 3 & 3 \\
0 & 1 & -5
\end{pmatrix}
= 
\begin{pmatrix}
4 & 3 & 3 \\
0 & 1 & -5
\end{pmatrix} \tag{4}
\]

Based on the matrix on the right side of equation (4), add the second row multiplied by \(-3\) to the first row.

\[
\begin{pmatrix}
4 & 3 + 1 \times (-3) & 3 + (-5) \times (-3) \\
0 & 1 & -5
\end{pmatrix}
= 
\begin{pmatrix}
4 & 0 & 18 \\
0 & 1 & -5
\end{pmatrix} \tag{5}
\]

Based on the matrix on the right side of equation (5), divide the first row by 4.

\[
\begin{pmatrix}
4/4 & 0 & 18/4 \\
0 & 1 & -5
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 9/2 \\
0 & 1 & -5
\end{pmatrix}
\]

So the solutions are \( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 9/2 \\ -5 \end{pmatrix} \).
Problem 2.
a. Use elementary row operations to solve the following system of equations. The answers are $x_1 = 1$, $x_2 = 4$, $x_3 = 3$.

$$\begin{bmatrix} -5 & 2 & 0 \\ 1 & -1 & -1 \\ -4 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 7 \end{bmatrix}$$

Create the augmented matrix $\tilde{A}$. That is,

$$\tilde{A} = (C \ c) = \begin{bmatrix} -5 & 2 & 0 & 3 \\ 1 & -1 & -1 & -6 \\ -4 & 2 & 1 & 7 \end{bmatrix}$$

Then use elementary row operations on matrix $\tilde{A}$ to create an identity matrix on the left side.

To begin with, based on matrix $\tilde{A}$, add the second row multiplied by 4 to the third row.

$$\begin{bmatrix} -5 & 2 & 0 & 3 \\ 1 & -1 & -1 & -6 \\ -4 + 1 \times 4 & 2 + (-1) \times 4 & 1 + (-1) \times 4 & 7 + (-6) \times 4 \end{bmatrix} = \begin{bmatrix} -5 & 2 & 0 & 3 \\ 1 & -1 & -1 & -6 \\ 0 & 0 & 3 & -17 \end{bmatrix} \tag{6}$$

Based on matrix on the right side of equation (6), add the first row multiplied by $1/5$ to the second row.

$$\begin{bmatrix} -5 & 2 & 0 & 3 \\ 0 & 1 + (-5) \times (1/5) & -1 + 2 \times (1/5) & -1 - 6 + 3 \times (1/5) \\ 0 & -2 & -3 & -17 \end{bmatrix} = \begin{bmatrix} -5 & 2 & 0 & 3 \\ 0 & -3/5 & -1 & -27/5 \\ 0 & -2 & -3 & -17 \end{bmatrix} \tag{7}$$

Based on matrix on the right side of equation (7), multiply the second row by $-5/3$.

$$\begin{bmatrix} -5 & 2 & 0 & 3 \\ 0 & -3/5 \times (-5/3) & -1 \times (-5/3) & -27/5 \times (-5/3) \\ 0 & -2 & -3 & -17 \end{bmatrix} = \begin{bmatrix} -5 & 2 & 0 & 3 \\ 0 & 1 & 5/3 & 9 \end{bmatrix} \tag{8}$$

Based on matrix on the right side of equation (8), add the second row multiplied by 2 to the third row.

$$\begin{bmatrix} -5 & 2 & 0 & 3 \\ 0 & 1 & 5/3 & 9 \\ 0 & -2 + 1 \times 2 & -3 + (5/3) \times 2 & -17 + 9 \times 2 \end{bmatrix} = \begin{bmatrix} -5 & 2 & 0 & 3 \\ 0 & 1 & 5/3 & 9 \end{bmatrix} \tag{9}$$

Based on matrix on the right side of equation (9), multiply the third row by 3.

$$\begin{bmatrix} -5 & 2 & 0 & 3 \\ 0 & 1 & 5/3 & 9 \\ 0 & 0 & 1 \times 3 & 1 \times 3 \end{bmatrix} = \begin{bmatrix} -5 & 2 & 0 & 3 \\ 0 & 1 & 5/3 & 9 \end{bmatrix} \tag{10}$$
Based on matrix on the right side of equation (10), add the third row multiplied by $-5/3$ to the second row.

$$
\begin{pmatrix}
-5 & 2 & 0 & 3 \\
0 & 1 & 5/3 + 1 \times (-5/3) & 9 + 3 \times (-5/3) \\
0 & 0 & 1 & 3
\end{pmatrix}
= 
\begin{pmatrix}
-5 & 2 & 0 & 3 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 3
\end{pmatrix}
$$

(11)

Based on matrix on the right side of equation (11), add the second row multiplied by $-2$ to the first row.

$$
\begin{pmatrix}
-5 & 2 + 1 \times (-2) & 0 & 3 + 4 \times (-2) \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 3
\end{pmatrix}
= 
\begin{pmatrix}
-5 & 0 & 0 & -5 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 3
\end{pmatrix}
$$

(12)

Based on matrix on the right side of equation (12), divide the first row by $-5$.

$$
\begin{pmatrix}
-5/(-5) & 0 & 0 & -5/(-5) \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 3
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 4 \\
0 & 0 & 1 & 3
\end{pmatrix}
$$

(13)

The solutions are

$$
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
4 \\
3
\end{pmatrix}
$$
b. Use elementary row operations to solve the following system of equations. Use elementary row operations to solve the following system of equations. The answers are \(x_1 = 4, x_2 = 2, x_3 = 3\).

\[
Dx = d
\]

\[
\begin{bmatrix}
1 & -4 & 2 \\
-3 & 13 & -7 \\
4 & -4 & -5
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
2 \\
-7 \\
-7
\end{bmatrix}
\]

Create the augmented matrix \(\tilde{A}\). That is,

\[
\tilde{A} = (D \ d) = 
\begin{bmatrix}
1 & -4 & 2 & 2 \\
-3 & 13 & -7 & -7 \\
4 + 1 \times (-4) & -4 - 4 \times (-4) & -5 + 2 \times (-4) & -7 + 2 \times (-4)
\end{bmatrix}
\]

Based on matrix on the right side of equation (14), add the first row multiplied by \(-4\) to the third row.

\[
\left(
\begin{array}{cccc}
1 & -4 & 2 & 2 \\
-3 & 13 & -7 & -7 \\
4 + 1 \times (-4) & -4 - 4 \times (-4) & -5 + 2 \times (-4) & -7 + 2 \times (-4)
\end{array}
\right)
= 
\left(
\begin{array}{cccc}
1 & -4 & 2 & 2 \\
-3 & 13 & -7 & -7 \\
0 & 12 & -13 & -15
\end{array}
\right)
\]

Based on matrix on the right side of equation (15), add the second row multiplied by \(-12\) to the third row.

\[
\left(
\begin{array}{cccc}
1 & -4 & 2 & 2 \\
-3 + 1 \times 3 & 13 + (-4) \times 3 & -7 + 2 \times 3 & -7 + 2 \times 3 \\
0 & 12 & -13 & -15
\end{array}
\right)
= 
\left(
\begin{array}{cccc}
1 & -4 & 2 & 2 \\
0 & 1 & -1 & -1 \\
0 & 0 & -1 & -3
\end{array}
\right)
\]

Based on matrix on the right side of equation (16), add the second row multiplied by \(-1\) to the first row.

\[
\left(
\begin{array}{cccc}
1 & -4 + 1 \times 4 & 2 + (-1) \times 4 & 2 + (-1) \times 4 \\
0 & 1 & -1 & -1 \\
0 & 0 & -1 & -3
\end{array}
\right)
= 
\left(
\begin{array}{cccc}
1 & 0 & -2 & -2 \\
0 & 1 & -1 & -1 \\
0 & 0 & -1 & -3
\end{array}
\right)
\]

Based on matrix on the right side of equation (17), multiply the third row by \(-1\).

\[
\left(
\begin{array}{cccc}
1 & 0 & -2 & -2 \\
0 & 1 & -1 & -1 \\
0 & 0 & -1 \times (-1) & -3 \times (-1)
\end{array}
\right)
= 
\left(
\begin{array}{cccc}
1 & 0 & -2 & -2 \\
0 & 1 & -1 & -1 \\
0 & 0 & 1 & 3
\end{array}
\right)
\]
Based on matrix on the right side of equation (18), add the third row to the second row.
\[
\begin{pmatrix}
1 & 0 & -2 & -2 \\
0 & 1 & -1+1 & -1+3 \\
0 & 0 & 1 & 3
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & -2 & -2 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 3
\end{pmatrix}
\]
(19)

Based on matrix on the right side of equation (19), add the third row multiplied by 2 to the first row.
\[
\begin{pmatrix}
1 & 0 & -2 + 1 \times 2 & -2 + 3 \times 2 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 3
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 4 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 3
\end{pmatrix}
\]
(20)

The solutions are \[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
= \begin{pmatrix}
4 \\
2 \\
3
\end{pmatrix}.
\]
Problem 3.
a. Use elementary row operations to find the inverse of the following matrix.

\[ A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \]

The answer is

\[ A^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \]

Augment matrix \( A \) with an \( 2 \times 2 \) identity matrix. That is, let

\[ \tilde{A} = (A \ I_{2 \times 2}) = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{pmatrix} \]

Then use elementary row operations on \( \tilde{A} \) to create an identity matrix on the left side. To begin with, based on \( \tilde{A} \), add the first row multiplied by \(-2\) to the second row.

\[ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 + 1 \times (-2) & 3 + 1 \times (-2) & 0 + 1 \times (-2) & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix} \] (21)

Based on the matrix on the right side of equation (21), subtract the second row from the first row.

\[ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \end{pmatrix} \] (22)

So the inverse of \( A \) is

\[ A^{-1} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \]
b. Use elementary row operations to find the inverse of the following matrix.

\[ E = \begin{pmatrix} 5 & 9 \\ 2 & 3 \end{pmatrix} \]

The answer is

\[ E^{-1} = \begin{pmatrix} -1 & 3 \\ 5/2 & -5/3 \end{pmatrix} \]

Augment matrix \( E \) with an \( 2 \times 2 \) identity matrix. That is, let

\[ \tilde{A} = \begin{pmatrix} A & I_{2x2} \end{pmatrix} = \begin{pmatrix} 5 & 9 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{pmatrix} \]

Then use elementary row operations on \( \tilde{A} \) to create an identity matrix on the left side.

To begin with, based on \( \tilde{A} \), add the first row multiplied by \(-2/5\) to the second row.

\[ \begin{pmatrix} 5 & 9 & 1 & 0 \\ 2 + 5 \times (-2/5) & 3 + 9 \times (-2/5) & 0 + 1 \times (-2/5) & 1 \end{pmatrix} = \begin{pmatrix} 5 & 9 & 1 & 0 \\ 0 & -3/5 & -2/5 & 1 \end{pmatrix} \quad (23) \]

Based on the matrix on the right side of equation (23), multiply the second row by \(-5/3\).

\[ \begin{pmatrix} 5 & 9 \\ 0 & -3/5 \times (-5/3) \end{pmatrix} = \begin{pmatrix} 5 & 9 \\ 0 & 1 \end{pmatrix} \quad (24) \]

Based on the matrix on the right side of equation (24), add the second row multiplied by \(-9\) to the first row.

\[ \begin{pmatrix} 5 & 9 - 1 \times 9 & 1 - (2/3) \times 9 & 0 - (-5/3) \times 9 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 & -5 & 15 \\ 0 & 1 & 2/3 & -5/3 \end{pmatrix} \quad (25) \]

Based on the matrix on the right side of equation (25), divide the first row by 5.

\[ \begin{pmatrix} 5/5 & 0 & -5/5 & 15/5 \\ 0 & 1 & 2/3 & -5/3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 2/3 & -5/3 \end{pmatrix} \quad (26) \]

So the inverse of \( E \) is

\[ E^{-1} = \begin{pmatrix} -1 & 3 \\ 5/2 & -5/3 \end{pmatrix} \]
Problem 4.

a. Use elementary row operations to find the inverse of the following matrix.

\[ F = \begin{bmatrix} 3 & 6 & -3 \\ -1 & -5/3 & 2 \\ -3 & -4 & 8 \end{bmatrix} \]

The answer is

\[ F^{-1} = \begin{bmatrix} 16/3 & 36 & -7 \\ -2 & -15 & 3 \\ 1 & 6 & -1 \end{bmatrix} \]

Augment matrix \( F \) with an \( 3 \times 3 \) identity matrix. That is, let

\[ \tilde{A} = [F \ I_{3\times3}] = \begin{bmatrix} 3 & 6 & -3 & 1 & 0 & 0 \\ -1 & -5/3 & 2 & 0 & 1 & 0 \\ -3 & -4 & 8 & 0 & 0 & 1 \end{bmatrix} \]

Then use elementary row operations on matrix \( \tilde{A} \) to create an identity matrix on the left side. To begin with, based on matrix \( \tilde{A} \), add the first row to the third row.

\[ \begin{bmatrix} 3 & 6 & -3 & 1 & 0 & 0 \\ -1 & -5/3 & 2 & 0 & 1 & 0 \\ 0 & 2 & 5 & 1 & 0 & 1 \end{bmatrix} \]

Based on matrix on the right side of equation (27), add the first row multiplied by 1/3 to the second row.

\[ \begin{bmatrix} 3 & 6 & -3 & 1 & 0 & 0 \\ 0 & 1/3 & 1 & 1/3 & 1 & 0 \\ 0 & 2 & 5 & 1 & 0 & 1 \end{bmatrix} \]

Based on matrix on the right side of equation (28), multiply the second row by 3.

\[ \begin{bmatrix} 3 & 6 & -3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 5 & 1 & 0 & 1 \end{bmatrix} \]
Based on matrix on the right side of equation (29), add the second row multiplied by $-2$ to the third row.

\[
\begin{bmatrix}
3 & 6 & -3 & 1 & 0 & 0 \\
0 & 1 & 3 & 1 & 3 & 0 \\
0 & 2 + 1 \times (-2) & 5 + 3 \times (-2) & 1 + 1 \times (-2) & 0 + 3 \times (-2) & 1
\end{bmatrix}
= \begin{bmatrix}
3 & 6 & -3 & 1 & 0 & 0 \\
0 & 1 & 3 & 1 & 3 & 0 \\
0 & 0 & -1 & -1 & -6 & 1
\end{bmatrix}
\]  

(30)

Based on matrix on the right side of equation (30), add the third row multiplied by 3 to the second row.

\[
\begin{bmatrix}
3 & 6 & -3 & 1 & 0 & 0 \\
0 & 1 & 3 + (-1) \times 3 & 1 + (-1) \times 3 & 3 + (-6) \times 3 & 0 + 1 \times 3 \\
0 & 0 & -1 & -1 & -6 & 1
\end{bmatrix}
= \begin{bmatrix}
3 & 6 & -3 & 1 & 0 & 0 \\
0 & 1 & 0 & -2 & -15 & 3 \\
0 & 0 & -1 & -1 & -6 & 1
\end{bmatrix}
\]  

(31)

Based on matrix on the right side of equation (31), multiply the third row by $-3$.

\[
\begin{bmatrix}
3 & 6 & -3 & 1 & 0 & 0 \\
0 & 1 & 0 & -2 & -15 & 3 \\
0 & 0 & -1 \times (-3) & -1 \times (-3) & -6 \times (-3) & 1 \times (-3)
\end{bmatrix}
= \begin{bmatrix}
3 & 6 & -3 & 1 & 0 & 0 \\
0 & 1 & 0 & -2 & -15 & 3 \\
0 & 0 & 1 & 1 & 6 & -1
\end{bmatrix}
\]  

(32)

Based on matrix on the right side of equation (32) add the third row multiplied by 3 to the first row.

\[
\begin{bmatrix}
3 & 6 & -3 + 1 \times 3 & 1 + 1 \times 3 & 0 + 6 \times 3 & 0 + (-1) \times 3 \\
0 & 1 & 0 & -2 & -15 & 3 \\
0 & 0 & 1 & 1 & 6 & -1
\end{bmatrix}
= \begin{bmatrix}
3 & 6 & 0 & 4 & 18 & -3 \\
0 & 1 & 0 & -2 & -15 & 3 \\
0 & 0 & 1 & 1 & 6 & -1
\end{bmatrix}
\]  

(33)
Based on matrix on the right side of equation (33), add the second row multiplied by $-6$ to the first row.

\[
\begin{bmatrix}
3 & 6 + 1 \times (-6) & 0 & 4 - 2 \times (-6) & 18 - 15 \times (-6) & -3 + 3 \times (-6)
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 1 & 1 & 6 & -1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
3 & 0 & 0 & 108 & -21
0 & 1 & 0 & -2 & -15 & 3
0 & 0 & 1 & 1 & 6 & -1
\end{bmatrix}
\]

(34)

Based on matrix on the right side of equation (34) divide the first row by 3.

\[
\begin{bmatrix}
3/3 & 0 & 0 & 16/3 & 108/3 & -21/3
0 & 1 & 0 & -2 & -15 & 3
0 & 0 & 1 & 1 & 6 & -1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 & 0 & 16/3 & 36 & -7
0 & 1 & 0 & -2 & -15 & 3
0 & 0 & 1 & 1 & 6 & -1
\end{bmatrix}
\]

(35)

So the inverse of $F$ is given by

\[
F^{-1} = \begin{bmatrix}
16/3 & 36 & -7
-2 & -15 & 3
1 & 6 & -1
\end{bmatrix}
\]
b. Use elementary row operations to find the inverse of the following matrix.

\[ D = \begin{bmatrix} 1 & -4 & 2 \\ -3 & 13 & -7 \\ 4 & -4 & -5 \end{bmatrix} \]

The answer is

\[ D^{-1} = \begin{bmatrix} 93 & 28 & -2 \\ 43 & 13 & -1 \\ 40 & 12 & -1 \end{bmatrix} \]

Augment matrix \( D \) with an \( 3 \times 3 \) identity matrix. That is, let

\[ \tilde{A} = [D \ I_{3 \times 3}] = \begin{bmatrix} 1 & -4 & 2 & 1 & 0 & 0 \\ -3 & 13 & -7 & 0 & 1 & 0 \\ 4 & -4 & -5 & 0 & 0 & 1 \end{bmatrix} \]

Then use elementary row operations on matrix \( \tilde{A} \) to create an identity matrix on the left side. To begin with, based on matrix \( \tilde{A} \), add the first row multiplied by \(-4\) to the third row.

\[ \begin{bmatrix} 1 & -4 & 2 & 1 & 0 & 0 \\ -3 & 13 & -7 & 0 & 1 & 0 \\ 4 + 1 \times (-4) & -4 + (-4) \times (-4) & -5 + 2 \times (-4) & 0 + 1 \times (-4) & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 2 & 1 & 0 & 0 \\ -3 & 13 & -7 & 0 & 1 & 0 \\ 0 & -2 & -13 & -4 & 0 & 1 \end{bmatrix} \quad (36) \]

Based on matrix on the right side of equation (36) add the first row multiplied by 3 to the second row.

\[ \begin{bmatrix} 1 & -4 & 2 & 1 & 0 & 0 \\ -3 + 1 \times 3 & 13 + (-4) \times 3 & -7 + 2 \times 3 & 0 + 1 \times 3 & 1 & 0 \\ 0 & 12 & -13 & -4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 3 & 1 & 0 \\ 0 & 12 & -13 & -4 & 0 & 1 \end{bmatrix} \quad (37) \]

Based on matrix on the right side of equation (37), add the second row multiplied by 4 to the first row.

\[ \begin{bmatrix} 1 & -4 + 1 \times 4 & 2 + (-1) \times 4 & 1 + 3 \times 4 & 0 + 1 \times 4 & 0 \\ 0 & 1 & -1 & 3 & 1 & 0 \\ 0 & 12 & -13 & -4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & 13 & 4 & 0 \\ 0 & 1 & -1 & 3 & 1 & 0 \\ 0 & 12 & -13 & -4 & 0 & 1 \end{bmatrix} \quad (38) \]
Based on matrix on the right side of equation (38), add the second row multiplied by $-12$ to the third row.

\[
\begin{bmatrix}
1 & 0 & -2 & 13 & 4 & 0 \\
0 & 1 & -1 & 3 & 1 & 0 \\
0 & 12 + 1 \times (-12) & -13 + (-1) \times (-12) & -4 + 3 \times (-12) & 0 + 1 \times (-12) & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 & -2 & 13 & 4 & 0 \\
0 & 1 & -1 & 3 & 1 & 0 \\
0 & 0 & -1 & 3 & 1 & 0
\end{bmatrix}
\]

(39)

Based on matrix on the right side of equation (39), multiply the third row by $-1$.

\[
\begin{bmatrix}
1 & 0 & -2 & 13 & 4 & 0 \\
0 & 1 & -1 & 3 & 1 & 0 \\
0 & 0 & -1 \times (-1) & -40 \times (-1) & -12 \times (-1) & 1 \times (-1)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 & -2 & 13 & 4 & 0 \\
0 & 1 & -1 & 3 & 1 & 0 \\
0 & 0 & 1 & 40 & 12 & -1
\end{bmatrix}
\]

(40)

Based on matrix on the right side of equation (40) add the third row multiplied by $2$ to the first row.

\[
\begin{bmatrix}
1 & 0 & -2 + 1 \times 2 & 13 + 40 \times 2 & 4 + 12 \times 2 & 0 + (-1) \times 2 \\
0 & 1 & -1 & 3 & 1 & 0 \\
0 & 0 & 1 & 40 & 12 & -1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 & 0 & 93 & 28 & -2 \\
0 & 1 & -1 & 3 & 1 & 0 \\
0 & 0 & 1 & 40 & 12 & -1
\end{bmatrix}
\]

(41)

Based on matrix on the right side of equation (41) add the third row to the second row.

\[
\begin{bmatrix}
1 & 0 & 0 & 93 & 28 & -2 \\
0 & 1 & -1 + 1 & 3 + 40 & 1 + 12 & 0 - 1 \\
0 & 0 & 1 & 40 & 12 & -1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 0 & 0 & 93 & 28 & -2 \\
0 & 1 & 0 & 43 & 13 & -1 \\
0 & 0 & 1 & 40 & 12 & -1
\end{bmatrix}
\]

(42)
Problem 5.

a. Use the inverse you found in problem 3a to solve the following equation.

\[ A \mathbf{x} = \mathbf{a} \]

\[
\begin{pmatrix}
1 & 1 \\
2 & 3
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
1
\end{pmatrix}
\]

The answer is

\[
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= 
\begin{pmatrix}
2 \\
-1
\end{pmatrix}
\]

Using the inverse of \( A \), the solution is given by

\[
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= A^{-1} \mathbf{a} = 
\begin{pmatrix}
3 & -1 \\
-2 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
3 \times 1 + (-1) \times 1 \\
-2 \times 1 + 1 \times 1
\end{pmatrix}
= 
\begin{pmatrix}
2 \\
-1
\end{pmatrix}
\]

b. Use the inverse you found in problem 3b to solve the following equation.

\[ E \mathbf{x} = \mathbf{c} \]

\[
\begin{pmatrix}
5 & 9 \\
2 & 3
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= 
\begin{pmatrix}
2 \\
-3
\end{pmatrix}
\]

The answer is

\[
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= 
\begin{pmatrix}
-11 \\
19
\end{pmatrix}
\]

Using the inverse of matrix \( E \), the solution is given by

\[
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
= E^{-1} \mathbf{c} = 
\begin{pmatrix}
-1 & 3 \\
\frac{2}{3} & -\frac{5}{3}
\end{pmatrix}
\begin{pmatrix}
2 \\
-3
\end{pmatrix}
\]

\[
= \begin{pmatrix}
-1 \times 2 + 3 \times (-3) \\
\frac{2}{3} \times 2 + \frac{-5}{3} \times (-3)
\end{pmatrix}
= 
\begin{pmatrix}
-11 \\
19
\end{pmatrix}
\]

\[
= 
\begin{pmatrix}
\frac{-11}{3}
\end{pmatrix}
\]
Problem 6. Use the inverse you found in problem 4a to solve the following equation.

\[ Fx = f \]

\[
\begin{bmatrix}
3 & 6 & -3 \\
-1 & -5/3 & 2 \\
-3 & -4 & 8
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
3 \\
-1 \\
2
\end{bmatrix}
\]

Use the inverse of \( F \), the solution is given by

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = F^{-1} f = 
\begin{bmatrix}
16/3 & 36 & -7 \\
-2 & -15 & 3 \\
1 & 6 & -1
\end{bmatrix}
\begin{bmatrix}
3 \\
-1 \\
2
\end{bmatrix}
= 
\begin{bmatrix}
(16/3) \times 3 + 36 \times (-1) + (-7) \times 2 \\
-2 \times 3 + (-15) \times (-1) + 3 \times 2 \\
1 \times 3 + 6 \times (-1) + (-1) \times 2
\end{bmatrix}
= 
\begin{bmatrix}
-34 \\
15 \\
-5
\end{bmatrix}
\]
Problem 7. Compute the determinants of the following matrices.

a.

\[
A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}
\]

\[\det[A] = 1 \times 3 - 1 \times 2 = 1\]

b.

\[
B = \begin{pmatrix} 4 & 3 \\ 2 & 2 \end{pmatrix}
\]

\[\det[B] = 4 \times 2 - 3 \times 2 = 2\]

c.

\[
C = \begin{pmatrix} -5 & 2 & 0 \\ 1 & -1 & -1 \\ -4 & 2 & 1 \end{pmatrix}
\]

\[\det[C] = 1\]

\[\det[C] = \begin{vmatrix} -5 & 2 & 0 \\ 1 & -1 & -1 \\ -4 & 2 & 1 \end{vmatrix} = -5 \times \begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix} - 2 \times \begin{vmatrix} 1 & -1 \\ -4 & 1 \end{vmatrix} = -5 \times 1 + 6 - (-1) \times (-4) = -5 + 6 = 1\]

d.

\[
D = \begin{pmatrix} 1 & -4 & 2 \\ -3 & 13 & -7 \\ 4 & -4 & -5 \end{pmatrix}
\]

\[\det[D] = -1\]

\[\det[D] = \begin{vmatrix} 1 & -4 & 2 \\ -3 & 13 & -7 \\ 4 & -4 & -5 \end{vmatrix} = 1 \times \begin{vmatrix} 13 & -7 \\ -4 & -5 \end{vmatrix} - (-4) \times \begin{vmatrix} -3 & -7 \\ 4 & -5 \end{vmatrix} + 2 \times \begin{vmatrix} -3 & 13 \\ 4 & -4 \end{vmatrix}
= -93 + 172 - 80 = -1\]
Problem 8. For each of the following problems, find the critical points. For each critical point state whether the function is at a relative maximum, relative minimum, or otherwise. Check to see if there are potential points of inflection at points other than critical points.

a. $f(x) = 6x^4 + \frac{52}{3}x^3 - 30x^2$.

The derivatives of $f(x) = 6x^4 + \frac{52}{3}x^3 - 30x^2$ are given by

$$f'(x) = 24x^3 + 52x^2 - 60x \quad (43)$$
$$f''(x) = 72x^2 + 104x - 60 \quad (44)$$
$$f'''(x) = 144x + 104 \quad (45)$$

Set the first derivative, equation (43), to be zero.

$$f'(x) = 24x^3 + 52x^2 - 60x = 0$$

$$\Rightarrow \quad x(6x^2 + 13x - 15) = 0$$

$$\Rightarrow \quad x(x + 3)(6x - 5) = 0$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = -3 \quad \text{or} \quad x = 5/6$$

Check the second derivative, equation (44), for $x = 0$, $x = -3$, and $x = 5/6$ respectively.

$$f''(0) = 72 \times 0^2 + 104 \times 0 - 60 = -60 < 0$$

$$f''(-3) = 72 \times (-3)^2 + 104 \times (-3) - 60 = 276 > 0$$

$$f''(5/6) = 72 \times (5/6)^2 + 104 \times (5/6) - 60 = 230/3 > 0$$

So $x = 0$ is a local maximum point; $x = -3$ and $x = 5/6$ are local maximum points.
Set the second derivative, equation (44), to be zero.
\[ f''(x) = 72x^2 + 104x - 60 = 0 \]
\[ \Rightarrow \quad 18x^2 + 26x - 15 = 0 \]
\[ \Rightarrow \quad x = \frac{-26 \pm \sqrt{26^2 - 4 \times 18 \times (-15)}}{36} \]
\[ \Rightarrow \quad x = \frac{-13 \pm \sqrt{439}}{18} \]

Since \( f^{(3)}(x) = 144x + 104 \) is zero only when \( x = -104/144 = 13/18, x = \frac{-13 \pm \sqrt{439}}{18} \) are inflection points.
b. \( f(x) = 4e^{-3x} \left( \frac{28}{9} - \frac{17x}{3} - 2x^2 \right) \)

**Figure 2.** \( f(x) = 4e^{-3x} \left( \frac{28}{9} - \frac{17x}{3} - 2x^2 \right) \)

**Figure 3.** A Close-up

Hints: The second derivative of the function will simplify to \(-72e^{-3x}x^2 - 108e^{-3x}x + 232e^{-3x}\). The real points of inflection (you still need to find them) are \( \frac{1}{12} \) \((-9 \pm \sqrt{545}) \). \( 9 \times 545 = 4905 \).

The derivatives of \( f(x) = 4e^{-3x} \left( \frac{28}{9} - \frac{17x}{3} - 2x^2 \right) \) are given by

\[
f'(x) = 4e^{-3x} \cdot (-3) \left( \frac{28}{9} - \frac{17x}{3} - 2x^2 \right) + 4e^{-3x} \left( -\frac{17}{3} - 4x \right)
= e^{-3x} \left( -60 + 52x + 24x^2 \right)
\]

\[
f''(x) = e^{-3x} \cdot (-3) \left( -60 + 52x + 24x^2 \right) + e^{-3x} (52 + 48x)
= e^{-3x} \left( 232 - 108x - 72x^2 \right)
\]

\[
f'''(x) = e^{-3x} \cdot (-3) \left( 232 - 108x - 72x^2 \right) + e^{-3x} (-108 - 144x)
= e^{-3x} \left( -804 + 180x + 216x^2 \right)
= 12e^{-3x} \left( -67 + 15x + 18x^2 \right)
\]

Set the first derivative, equation (46), to zero.

\[
f'(x) = e^{-3x} \left( -60 + 52x + 24x^2 \right) = 0
\]

\[
\Rightarrow \quad -60 + 52x + 24x^2 = 0
\]

\[
\Rightarrow \quad 4(-15 + 13x + 6x^2) = 0
\]

\[
\Rightarrow \quad 4(x + 3)(6x - 5) = 0
\]

\[
\Rightarrow \quad x = -3 \quad \text{or} \quad x = 5/6
\]

Check the second derivative, equation (47), for \( x = -3 \) and \( x = 5/6 \) respectively.

\[
f''(-3) = e^{-3x(-3)} \left( 232 - 108 \times (-3) - 72 \times (-3)^2 \right)
= e^9 \times (-92) < 0
\]

\[
f''(5/6) = e^{-3x(5/6)} \left( 232 - 108 \times (5/6) - 72 \times (5/6)^2 \right)
= e^{-5/2} \times 92 > 0
\]
So $x = -3$ is a local maximum point; $x = 5/6$ is a local minimum point.

Set the second derivative, equation (47), to be zero.

$$f''(x) = e^{-3x} (232 - 108x - 72x^2) = 0$$

$$\Rightarrow \quad 232 - 108x - 72x^2 = 0$$

$$\Rightarrow \quad 58 - 27x - 18x^2 = 0$$

$$\Rightarrow \quad x = \frac{27 \pm \sqrt{27^2 - 4 \times (-18) \times 58}}{-36}$$

$$\Rightarrow \quad x = \frac{9 \pm \sqrt{545}}{-12}$$

Also set third derivative, equation (48), to be zero.

$$f^{(3)}(x) = 12e^{-3x} (-67 + 15x + 18x^2) = 0$$

$$\Rightarrow \quad -67 + 15x + 18x^2 = 0$$

$$\Rightarrow \quad x = \frac{-15 \pm \sqrt{15^2 - 4 \times 18 \times (-67)}}{36}$$

$$\Rightarrow \quad x = \frac{-5 \pm \sqrt{761}}{12}$$

As a result, when $x = \frac{9 \pm \sqrt{545}}{-12} \neq \frac{-5 \pm \sqrt{761}}{12}$, the third derivatives are not equal to zero.

Thus, $x = \frac{9 \pm \sqrt{545}}{-12}$ are two inflection points.
c. \( f(x) = x^4 + \frac{8x^3}{3} - \frac{15x^2}{2} - 9x \)

**Figure 4.** \( f(x) = x^4 + \frac{8x^3}{3} - \frac{15x^2}{2} - 9x \)

The inflection points are \( \frac{1}{6} \left(-4 \pm \sqrt{61}\right) \).

The derivatives of \( f(x) = x^4 + \frac{8x^3}{3} - \frac{15x^2}{2} - 9x \) are given by
\[
\begin{align*}
f'(x) &= 4x^3 + 8x^2 - 15x - 9 \\
f''(x) &= 12x^2 + 16x - 15 \\
f'''(x) &= 24x + 16
\end{align*}
\]

Set the first derivative, equation (49), to be zero.
\[
f'(x) = 4x^3 + 8x^2 - 15x - 9 = 0
\]

By guess, \( x = -3 \) is a solution of equation \( f'(x) = 0 \). Then we try to factorize \( f'(x) = 4x^3 + 8x^2 - 15x - 9 = 0 \) by \( (x + 3) \).

\[
\begin{array}{c|cccc}
  & 4x^2 & -4x & 3 \\
\hline
x+3 & 4x^3 & +8x^2 & -15x & -9 \\
& -4x^3 & -12x^2 & \\
& & -4x^2 & -15x & \\
& & & 4x^2 & +12x \\
& & & -3x & -9 & \\
& & & & 3x & +9 & \\
& & & & & 0
\end{array}
\]

Then, equation (52) can be easily solved.
\[
f'(x) = 4x^3 + 8x^2 - 15x - 9 = 0 \\
\Rightarrow (x+3)(4x^2 - 4x - 3) = 0 \\
\Rightarrow (x+3)(2x+1)(2x-3) = 0 \\
\Rightarrow x = -3 \text{ or } x = -1/2 \text{ or } x = 3/2
Check the second derivative, equation (50), for \( x = -3, x = -1/2, \) and \( x = 3/2. \)

\[
f''(-3) = 12 \times (-3)^2 + 16 \times (-3) - 15 = 45 > 0
\]

\[
f''(-1/2) = 12 \times (-1/2)^2 + 16 \times (-1/2) - 15 = -20 < 0
\]

\[
f''(3/2) = 12 \times (3/2)^2 + 16 \times (3/2) - 15 = 36 > 0
\]

So \( x = -3 \) and \( x = 3/2 \) are local minimum points; \( x = -1/2 \) is a local minimum point.

Set the second derivative, equation (50), to be zero.

\[
f''(x) = 12x^2 + 16x - 15 = 0
\]

\[
\Rightarrow \quad x = \frac{-16 \pm \sqrt{16^2 - 4 \times 12 \times (-15)}}{24}
\]

\[
\Rightarrow \quad x = \frac{-4 \pm \sqrt{61}}{6}
\]

Since \( f^{(3)}(x) = 24x + 16 \) is zero only when \( x = -16/24 = -2/3, \) \( x = \frac{-4 \pm \sqrt{61}}{6} \) are inflection points.