ECONOMICS 207
SPRING 2008
PROBLEM SET 10
KEY

For this laboratory exercise, consider the following matrices and vectors. You may want to tear this page off so it is easy to view.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} -5 & 2 & 0 \\ 1 & -1 & -1 \\ -4 & 2 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & -4 & 2 \\ -3 & 13 & -7 \\ 4 & -4 & -5 \end{bmatrix}, \quad E = \begin{bmatrix} 5 & 9 \\ 2 & 3 \end{bmatrix}, \quad F = \begin{bmatrix} 3 & 6 & -3 \\ -1 & -5/3 & 2 \\ -3 & -4 & 8 \end{bmatrix}$$

$$a = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad c = \begin{bmatrix} 3 \\ -6 \\ 7 \end{bmatrix} \quad d = \begin{bmatrix} 2 \\ -7 \\ -7 \end{bmatrix}, \quad e = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad f = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

Date: August 25, 2008.

Problem 1.

a. Use elementary row operations to solve the following system of equations.

$$Ax = a$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Create the augmented matrix $\tilde{A}$. That is,

$$\tilde{A} = \begin{pmatrix} A & a \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \end{pmatrix}$$

Then use elementary row operations on matrix $\tilde{A}$ to create an identity matrix on the left side. To begin with, based on $\tilde{A}$, add the first row multiplied by -2 to the second row.

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 + 1 \times (-2) & 3 + 1 \times (-2) & 1 + 1 \times (-2) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \tag{1}$$

Based on the matrix on the right side of equation (1), subtract the second row from the first row.

$$\begin{pmatrix} 1 & 1 - 1 & 1 - (-1) \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \tag{2}$$

So the solutions are $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

b. Use elementary row operations to solve the following system of equations.

$$\begin{pmatrix} 4 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

Create the augmented matrix $\tilde{A}$. That is,

$$\tilde{A} = \begin{pmatrix} 4 & 3 & 3 \\ 2 & 2 & -1 \end{pmatrix}$$

Then use elementary row operations on matrix $\tilde{A}$ to create an identity matrix on the left side. To begin with, based on $\tilde{A}$, add the first row multiplied by $-1/2$ to the second row.

$$\begin{pmatrix} 4 & 3 & 3 \\ 2 + 4 \times (-1/2) & 2 + 3 \times (-1/2) & -1 + 3 \times (-1/2) \end{pmatrix} = \begin{pmatrix} 4 & 3 & 3 \\ 0 & \frac{1}{2} & -\frac{5}{2} \end{pmatrix} \tag{3}$$

Based on the matrix on the right side of equation (3), multiply the second row by 2.

$$\begin{pmatrix} 4 & 3 & 3 \\ 0 & \frac{1}{2} \times 2 & -\frac{5}{2} \times 2 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 3 \\ 0 & 1 & -5 \end{pmatrix} \tag{4}$$

Based on the matrix on the right side of equation (4), add the second row multiplied by -3 to the first row.

$$\begin{pmatrix} 4 & 3 + 1 \times (-3) & 3 + (-5) \times (-3) \\ 0 & 1 & -5 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 18 \\ 0 & 1 & -5 \end{pmatrix} \tag{5}$$

Based on the matrix on the right side of equation (5), divide the first row by 4.

$$\begin{pmatrix} 4/4 & 0 & 18/4 \\ 0 & 1 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 9/2 \\ 0 & 1 & -5 \end{pmatrix}$$

So the solutions are $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 9/2 \\ -5 \end{pmatrix}$.

Problem 2.

a. Use elementary row operations to solve the following system of equations. The answers are $x_1 = 1$, $x_2 = 4$, $x_3 = 3$.

$$Cx = c$$

$$\begin{bmatrix} -5 & 2 & 0 \\ 1 & -1 & -1 \\ -4 & 2 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ 7 \end{pmatrix}$$

Create the augmented matrix $\tilde{A}$. That is,

$$\tilde{A} = \begin{pmatrix} C & c \end{pmatrix} = \begin{pmatrix} -5 & 2 & 0 & 3 \\ 1 & -1 & -1 & -6 \\ -4 & 2 & 1 & 7 \end{pmatrix}$$

Then use elementary row operations on matrix $\tilde{A}$ to create an identity matrix on the left side. To begin with, based on matrix $\tilde{A}$, add the second row multiplied by 4 to the third row.

$$\begin{pmatrix} -5 & 2 & 0 & 3 \\ 1 & -1 & -1 & -6 \\ -4 + 1 \times 4 & 2 + (-1) \times 4 & 1 + (-1) \times 4 & 7 + (-6) \times 4 \end{pmatrix} = \begin{pmatrix} -5 & 2 & 0 & 3 \\ 1 & -1 & -1 & -6 \\ 0 & -2 & -3 & -17 \end{pmatrix} \quad (6)$$

Based on matrix on the right side of equation (6), add the first row multiplied by 1/5 to the second row.

$$\begin{pmatrix} -5 & 2 & 0 & 3 \\ 1 + (-5) \times (1/5) & -1 + 2 \times (1/5) & -1 & -6 + 3 \times (1/5) \\ 0 & -2 & -3 & -17 \end{pmatrix} = \begin{pmatrix} -5 & 2 & 0 & 3 \\ 0 & -3/5 & -1 & -27/5 \\ 0 & -2 & -3 & -17 \end{pmatrix} \quad (7)$$

Based on matrix on the right side of equation (7), multiply the second row by $-5/3$.

$$\begin{pmatrix} -5 & 2 & 0 & 3 \\ 0 & -3/5 \times (-5/3) & -1 \times (-5/3) & -27/5 \times (-5/3) \\ 0 & -2 & -3 & -17 \end{pmatrix} = \begin{pmatrix} -5 & 2 & 0 & 3 \\ 0 & 1 & 5/3 & 9 \\ 0 & -2 & -3 & -17 \end{pmatrix} \quad (8)$$

Based on matrix on the right side of equation (8), add the second row multiplied by 2 to the third row.

$$\begin{pmatrix} -5 & 2 & 0 & 3 \\ 0 & 1 & 5/3 & 9 \\ 0 & -2 + 1 \times 2 & -3 + (5/3) \times 2 & -17 + 9 \times 2 \end{pmatrix} = \begin{pmatrix} -5 & 2 & 0 & 3 \\ 0 & 1 & 5/3 & 9 \\ 0 & 0 & 1/3 & 1 \end{pmatrix} \quad (9)$$

Based on matrix on the right side of equation (9), multiply the third row by 3.

$$\begin{pmatrix} -5 & 2 & 0 & 3 \\ 0 & 1 & 5/3 & 9 \\ 0 & 0 & 1/3 \times 3 & 1 \times 3 \end{pmatrix} = \begin{pmatrix} -5 & 2 & 0 & 3 \\ 0 & 1 & 5/3 & 9 \\ 0 & 0 & 1 & 3 \end{pmatrix} \quad (10)$$

Based on matrix on the right side of equation (10), add the third row multiplied by $-5/3$ to the second row.

$$\begin{pmatrix} -5 & 2 & 0 & 3 \\ 0 & 1 & 5/3 + 1 \times (-5/3) & 9 + 3 \times (-5/3) \\ 0 & 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} -5 & 2 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix} \tag{11}$$

Based on matrix on the right side of equation (11), add the second row multiplied by -2 to the first row.

$$\begin{pmatrix} -5 & 2 + 1 \times (-2) & 0 & 3 + 4 \times (-2) \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} -5 & 0 & 0 & -5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix} \tag{12}$$

Based on matrix on the right side of equation (12), divide the first row by -5.

$$\begin{pmatrix} -5/(-5) & 0 & 0 & -5/(-5) \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix} \tag{13}$$

The solutions are $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$.

b. Use elementary row operations to solve the following system of equations. Use elementary row operations to solve the following system of equations. The answers are $x_1 = 4$, $x_2 = 2$, $x_3 = 3$.

$$Dx = d$$

$$\begin{bmatrix} 1 & -4 & 2 \\ -3 & 13 & -7 \\ 4 & -4 & -5 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -7 \\ -7 \end{pmatrix}$$

Create the augmented matrix $\tilde{A}$. That is,

$$\tilde{A} = \begin{pmatrix} D & d \end{pmatrix} = \begin{pmatrix} 1 & -4 & 2 & 2 \\ -3 & 13 & -7 & -7 \\ 4 & -4 & -5 & -7 \end{pmatrix}$$

Then use elementary row operations on matrix $\tilde{A}$ to create an identity matrix on the left side. To begin with, based on matrix $\tilde{A}$, add the first row multiplied by -4 to the third row.

$$\begin{pmatrix} 1 & -4 & 2 & 2 \\ -3 & 13 & -7 & -7 \\ 4+1\times(-4) & -4-4\times(-4) & -5+2\times(-4) & -7+2\times(-4) \end{pmatrix} = \begin{pmatrix} 1 & -4 & 2 & 2 \\ -3 & 13 & -7 & -7 \\ 0 & 12 & -13 & -15 \end{pmatrix} \tag{14}$$

Based on matrix on the right side of equation (14), add the first row multiplied by 3 to the second row.

$$\begin{pmatrix} 1 & -4 & 2 & 2 \\ -3+1\times 3 & 13+(-4)\times 3 & -7+2\times 3 & -7+2\times 3 \\ 0 & 12 & -13 & -15 \end{pmatrix} = \begin{pmatrix} 1 & -4 & 2 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 12 & -13 & -15 \end{pmatrix} \tag{15}$$

Based on matrix on the right side of equation (15), add the second row multiplied by -12 to the third row.

$$\begin{pmatrix} 1 & -4 & 2 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 12+1\times(-12) & -13+(-1)\times(-12) & -15-1\times(-12) \end{pmatrix} = \begin{pmatrix} 1 & -4 & 2 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & -3 \end{pmatrix} \tag{16}$$

Based on matrix on the right side of equation (16), add the second row multiplied by 4 to the first row.

$$\begin{pmatrix} 1 & -4+1\times 4 & 2+(-1)\times 4 & 2+(-1)\times 4 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & -3 \end{pmatrix} \tag{17}$$

Based on matrix on the right side of equation (17), multiply the third row by -1.

$$\begin{pmatrix} 1 & 0 & -2 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -1\times(-1) & -3\times(-1) \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix} \tag{18}$$

Based on matrix on the right side of equation (18), add the third row to the second row.

$$\begin{pmatrix} 1 & 0 & -2 & -2 \\ 0 & 1 & -1+1 & -1+3 \\ 0 & 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \tag{19}$$

Based on matrix on the right side of equation (19), add the third row multiplied by 2 to the first row.

$$\begin{pmatrix} 1 & 0 & -2+1\times 2 & -2+3\times 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \tag{20}$$

The solutions are $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$.

Problem 3.

a. Use elementary row operations to find the inverse of the following matrix.

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$$

The answer is

$$A^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$

Augment matrix A with an 2×2 identity matrix. That is, let

$$\tilde{A} = \begin{pmatrix} A & I_{2\times 2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{pmatrix}$$

Then use elementary row operations on $\tilde{A}$ to create an identity matrix on the left side.
To begin with, based on $\tilde{A}$, add the first row multiplied by -2 to the second row.

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 + 1 \times (-2) & 3 + 1 \times (-2) & 0 + 1 \times (-2) & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix} \tag{21}$$

Based on the matrix on the right side of equation (21), subtract the second row from the first row.

$$\begin{pmatrix} 1 & 1 - 1 & 1 - (-2) & 0 - 1 \\ 0 & 1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \end{pmatrix} \tag{22}$$

So the inverse of A is

$$A^{-1} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$$

b. Use elementary row operations to find the inverse of the following matrix.

$$E = \begin{pmatrix} 5 & 9 \\ 2 & 3 \end{pmatrix}$$

The answer is

$$E^{-1} = \begin{bmatrix} -1 & 3 \\ \frac{2}{3} & -\frac{5}{3} \end{bmatrix}$$

Augment matrix E with an 2×2 identity matrix. That is, let

$$\tilde{A} = \begin{pmatrix} A & I_{2\times 2} \end{pmatrix} = \begin{pmatrix} 5 & 9 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{pmatrix}$$

Then use elementary row operations on $\tilde{A}$ to create an identity matrix on the left side. To begin with, based on $\tilde{A}$, add the first row multiplied by $-2/5$ to the second row.

$$\begin{pmatrix} 5 & 9 & 1 & 0 \\ 2+5\times(-2/5) & 3+9\times(-2/5) & 0+1\times(-2/5) & 1 \end{pmatrix} = \begin{pmatrix} 5 & 9 & 1 & 0 \\ 0 & -3/5 & -2/5 & 1 \end{pmatrix} \tag{23}$$

Based on the matrix on the right side of equation (23), multiply the second row by $-5/3$.

$$\begin{pmatrix} 5 & 9 & 1 & 0 \\ 0 & -3/5\times(-5/3) & -2/5\times(-5/3) & 1\times(-5/3) \end{pmatrix} = \begin{pmatrix} 5 & 9 & 1 & 0 \\ 0 & 1 & 2/3 & -5/3 \end{pmatrix} \tag{24}$$

Based on the matrix on the right side of equation (24), add the second row multiplied by -9 to the first row.

$$\begin{pmatrix} 5 & 9-1\times 9 & 1-(2/3)\times 9 & 0-(-5/3)\times 9 \\ 0 & 1 & 2/3 & -5/3 \end{pmatrix} = \begin{pmatrix} 5 & 0 & -5 & 15 \\ 0 & 1 & 2/3 & -5/3 \end{pmatrix} \tag{25}$$

Based on the matrix on the right side of equation (25), divide the first row by 5.

$$\begin{pmatrix} 5/5 & 0 & -5/5 & 15/5 \\ 0 & 1 & 2/3 & -5/3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & 2/3 & -5/3 \end{pmatrix} \tag{26}$$

So the inverse of E is

$$E^{-1} = \begin{pmatrix} -1 & 3 \\ \frac{2}{3} & -\frac{5}{3} \end{pmatrix}$$

Problem 4.

a. Use elementary row operations to find the inverse of the following matrix.

$$F = \begin{bmatrix} 3 & 6 & -3 \\ -1 & -5/3 & 2 \\ -3 & -4 & 8 \end{bmatrix}$$

The answer is

$$F^{-1} = \begin{bmatrix} \frac{16}{3} & 36 & -7 \\ -2 & -15 & 3 \\ 1 & 6 & -1 \end{bmatrix}$$

Augment matrix F with an 3×3 identity matrix. That is, let

$$\tilde{A} = \begin{bmatrix} F & I_{3\times 3} \end{bmatrix} = \begin{bmatrix} 3 & 6 & -3 & 1 & 0 & 0 \\ -1 & -5/3 & 2 & 0 & 1 & 0 \\ -3 & -4 & 8 & 0 & 0 & 1 \end{bmatrix}$$

Then use elementary row operations on matrix $\tilde{A}$ to create an identity matrix on the left side. To begin with, based on matrix $\tilde{A}$, add the first row to the third row.

$$\begin{bmatrix} 3 & 6 & -3 & 1 & 0 & 0 \\ -1 & -5/3 & 2 & 0 & 1 & 0 \\ -3+3 & -4+6 & 8-3 & 0+1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & -3 & 1 & 0 & 0 \\ -1 & -5/3 & 2 & 0 & 1 & 0 \\ 0 & 2 & 5 & 1 & 0 & 1 \end{bmatrix} \tag{27}$$

Based on matrix on the right side of equation (27), add the first row multiplied by 1/3 to the second row.

$$\begin{bmatrix} 3 & 6 & -3 & 1 & 0 & 0 \\ -1+3\times(1/3) & -5/3+6\times(1/3) & 2+(-3)\times(1/3) & 0+1\times(1/3) & 1 & 0 \\ 0 & 2 & 5 & 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 & -3 & 1 & 0 & 0 \\ 0 & 1/3 & 1 & 1/3 & 1 & 0 \\ 0 & 2 & 5 & 1 & 0 & 1 \end{bmatrix} \tag{28}$$

Based on matrix on the right side of equation (28), multiply the second row by 3.

$$\begin{bmatrix} 3 & 6 & -3 & 1 & 0 & 0 \\ 0 & 1/3\times 3 & 1\times 3 & 1/3\times 3 & 1\times 3 & 0 \\ 0 & 2 & 5 & 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 & -3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 3 & 0 \\ 0 & 2 & 5 & 1 & 0 & 1 \end{bmatrix} \tag{29}$$

Based on matrix on the right side of equation (29), add the second row multiplied by −2 to the third row.

$$\begin{bmatrix} 3 & 6 & -3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 3 & 0 \\ 0 & 2+1\times(-2) & 5+3\times(-2) & 1+1\times(-2) & 0+3\times(-2) & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 6 & -3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 3 & 0 \\ 0 & 0 & -1 & -1 & -6 & 1 \end{bmatrix} \tag{30}$$

Based on matrix on the right side of equation (30), add the third row multiplied by 3 to the second row.

$$\begin{bmatrix} 3 & 6 & -3 & 1 & 0 & 0 \\ 0 & 1 & 3+(-1)\times 3 & 1+(-1)\times 3 & 3+(-6)\times 3 & 0+1\times 3 \\ 0 & 0 & -1 & -1 & -6 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 6 & -3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & -15 & 3 \\ 0 & 0 & -1 & -1 & -6 & 1 \end{bmatrix} \tag{31}$$

Based on matrix on the right side of equation (31), multiply the third row by −3.

$$\begin{bmatrix} 3 & 6 & -3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & -15 & 3 \\ 0 & 0 & -1\times(-3) & -1\times(-3) & -6\times(-3) & 1\times(-3) \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 6 & -3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & -15 & 3 \\ 0 & 0 & 1 & 1 & 6 & -1 \end{bmatrix} \tag{32}$$

Based on matrix on the right side of equation (32) add the third row multiplied by 3 to the first row.

$$\begin{bmatrix} 3 & 6 & -3+1\times 3 & 1+1\times 3 & 0+6\times 3 & 0+(-1)\times 3 \\ 0 & 1 & 0 & -2 & -15 & 3 \\ 0 & 0 & 1 & 1 & 6 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 6 & 0 & 4 & 18 & -3 \\ 0 & 1 & 0 & -2 & -15 & 3 \\ 0 & 0 & 1 & 1 & 6 & -1 \end{bmatrix} \tag{33}$$

Based on matrix on the right side of equation (33), add the second row multiplied by -6 to the first row.

$$\begin{bmatrix} 3 & 6+1\times(-6) & 0 & 4-2\times(-6) & 18-15\times(-6) & -3+3\times(-6) \\ 0 & 1 & 0 & -2 & -15 & 3 \\ 0 & 0 & 1 & 1 & 6 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 & 16 & 108 & -21 \\ 0 & 1 & 0 & -2 & -15 & 3 \\ 0 & 0 & 1 & 1 & 6 & -1 \end{bmatrix} \tag{34}$$

Based on matrix on the right side of equation (34) divide the first row by 3.

$$\begin{bmatrix} 3/3 & 0 & 0 & 16/3 & 108/3 & -21/3 \\ 0 & 1 & 0 & -2 & -15 & 3 \\ 0 & 0 & 1 & 1 & 6 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 16/3 & 36 & -7 \\ 0 & 1 & 0 & -2 & -15 & 3 \\ 0 & 0 & 1 & 1 & 6 & -1 \end{bmatrix} \tag{35}$$

So the inverse of F is given by

$$F^{-1} = \begin{bmatrix} 16/3 & 36 & -7 \\ -2 & -15 & 3 \\ 1 & 6 & -1 \end{bmatrix}$$

b. Use elementary row operations to find the inverse of the following matrix.

$$D = \begin{bmatrix} 1 & -4 & 2 \\ -3 & 13 & -7 \\ 4 & -4 & -5 \end{bmatrix}$$

The answer is

$$D^{-1} = \begin{bmatrix} 93 & 28 & -2 \\ 43 & 13 & -1 \\ 40 & 12 & -1 \end{bmatrix}$$

Augment matrix D with an 3×3 identity matrix. That is, let

$$\tilde{A} = \begin{bmatrix} D & I_{3\times3} \end{bmatrix} = \begin{bmatrix} 1 & -4 & 2 & 1 & 0 & 0 \\ -3 & 13 & -7 & 0 & 1 & 0 \\ 4 & -4 & -5 & 0 & 0 & 1 \end{bmatrix}$$

Then use elementary row operations on matrix $\tilde{A}$ to create an identity matrix on the left side. To begin with, based on matrix $\tilde{A}$, add the first row multiplied by -4 to the third row.

$$\begin{bmatrix} 1 & -4 & 2 & 1 & 0 & 0 \\ -3 & 13 & -7 & 0 & 1 & 0 \\ 4+1\times(-4) & -4+(-4)\times(-4) & -5+2\times(-4) & 0+1\times(-4) & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -4 & 2 & 1 & 0 & 0 \\ -3 & 13 & -7 & 0 & 1 & 0 \\ 0 & -2 & -13 & -4 & 0 & 1 \end{bmatrix} \tag{36}$$

Based on matrix on the right side of equation (36) add the first row multiplied by 3 to the second row.

$$\begin{bmatrix} 1 & -4 & 2 & 1 & 0 & 0 \\ -3+1\times 3 & 13+(-4)\times 3 & -7+2\times 3 & 0+1\times 3 & 1 & 0 \\ 0 & 12 & -13 & -4 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -4 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 3 & 1 & 0 \\ 0 & 12 & -13 & -4 & 0 & 1 \end{bmatrix} \tag{37}$$

Based on matrix on the right side of equation (37), add the second row multiplied by 4 to the first row.

$$\begin{bmatrix} 1 & -4+1\times 4 & 2+(-1)\times 4 & 1+3\times 4 & 0+1\times 4 & 0 \\ 0 & 1 & -1 & 3 & 1 & 0 \\ 0 & 12 & -13 & -4 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -2 & 13 & 4 & 0 \\ 0 & 1 & -1 & 3 & 1 & 0 \\ 0 & 12 & -13 & -4 & 0 & 1 \end{bmatrix} \tag{38}$$

Based on matrix on the right side of equation (38), add the second row multiplied by −12 to the third row.

$$\begin{bmatrix} 1 & 0 & -2 & 13 & 4 & 0 \\ 0 & 1 & -1 & 3 & 1 & 0 \\ 0 & 12+1\times(-12) & -13+(-1)\times(-12) & -4+3\times(-12) & 0+1\times(-12) & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -2 & 13 & 4 & 0 \\ 0 & 1 & -1 & 3 & 1 & 0 \\ 0 & 0 & -1 & -40 & -12 & 1 \end{bmatrix} \tag{39}$$

Based on matrix on the right side of equation (39), multiply the third row by −1.

$$\begin{bmatrix} 1 & 0 & -2 & 13 & 4 & 0 \\ 0 & 1 & -1 & 3 & 1 & 0 \\ 0 & 0 & -1\times(-1) & -40\times(-1) & -12\times(-1) & 1\times(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -2 & 13 & 4 & 0 \\ 0 & 1 & -1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 40 & 12 & -1 \end{bmatrix} \tag{40}$$

Based on matrix on the right side of equation (40) add the third row multiplied by 2 to the first row.

$$\begin{bmatrix} 1 & 0 & -2+1\times 2 & 13+40\times 2 & 4+12\times 2 & 0+(-1)\times 2 \\ 0 & 1 & -1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 40 & 12 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 93 & 28 & -2 \\ 0 & 1 & -1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 40 & 12 & -1 \end{bmatrix} \tag{41}$$

Based on matrix on the right side of equation (41) add the third row to the second row.

$$\begin{bmatrix} 1 & 0 & 0 & 93 & 28 & -2 \\ 0 & 1 & -1+1 & 3+40 & 1+12 & 0-1 \\ 0 & 0 & 1 & 40 & 12 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 93 & 28 & -2 \\ 0 & 1 & 0 & 43 & 13 & -1 \\ 0 & 0 & 1 & 40 & 12 & -1 \end{bmatrix} \tag{42}$$

Problem 5.

a. Use the inverse you found in problem 3a to solve the following equation.

$$Ax = a$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The answer is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Using the inverse of A, the solution is given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A^{-1}a = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \times 1 + (-1) \times 1 \\ -2 \times 1 + 1 \times 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

b. Use the inverse you found in problem 3b to solve the following equation.

$$Ex = e$$

$$\begin{pmatrix} 5 & 9 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

The answer is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -11 \\ \frac{19}{3} \end{pmatrix}$$

Using the inverse of matrix E, the solution is given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = E^{-1}c = \begin{pmatrix} -1 & 3 \\ \frac{2}{3} & -\frac{5}{3} \end{pmatrix} \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \times 2 + 3 \times (-3) \\ \frac{2}{3} \times 2 + \frac{-5}{3} \times (-3) \end{pmatrix}$$

$$= \begin{pmatrix} -11 \\ \frac{19}{3} \end{pmatrix}$$

Problem 6. Use the inverse you found in problem 4a to solve the following equation.

$$Fx = f$$

$$\begin{bmatrix} 3 & 6 & -3 \\ -1 & -5/3 & 2 \\ -3 & -4 & 8 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

Use the inverse of F, the solution is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = F^{-1}f = \begin{pmatrix} 16/3 & 36 & -7 \\ -2 & -15 & 3 \\ 1 & 6 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} (16/3) \times 3 + 36 \times (-1) + (-7) \times 2 \\ -2 \times 3 + (-15) \times (-1) + 3 \times 2 \\ 1 \times 3 + 6 \times (-1) + (-1) \times 2 \end{pmatrix}$$

$$= \begin{pmatrix} -34 \\ 15 \\ -5 \end{pmatrix}$$

Problem 7. Compute the determinants of the following matrices.

a.

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$$

$\det[A] = 1 \times 3 - 1 \times 2 = 1$

b.

$$B = \begin{pmatrix} 4 & 3 \\ 2 & 2 \end{pmatrix}$$

$\det[B] = 4 \times 2 - 3 \times 2 = 2$

c.

$$C = \begin{pmatrix} -5 & 2 & 0 \\ 1 & -1 & -1 \\ -4 & 2 & 1 \end{pmatrix}$$

$$\det[C] = 1$$

$$\det[C] = \begin{vmatrix} -5 & 2 & 0 \\ 1 & -1 & -1 \\ -4 & 2 & 1 \end{vmatrix} = -5 \times \begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix} - 2 \times \begin{vmatrix} 1 & -1 \\ -4 & 1 \end{vmatrix}$$
$$= -5 \times (1 \times 2 - (-1) \times (-4)) - 2 \times (1 \times 1 - (-1) \times (-4)) = -5 + 6 = 1$$

d.

$$D = \begin{pmatrix} 1 & -4 & 2 \\ -3 & 13 & -7 \\ 4 & -4 & -5 \end{pmatrix}$$

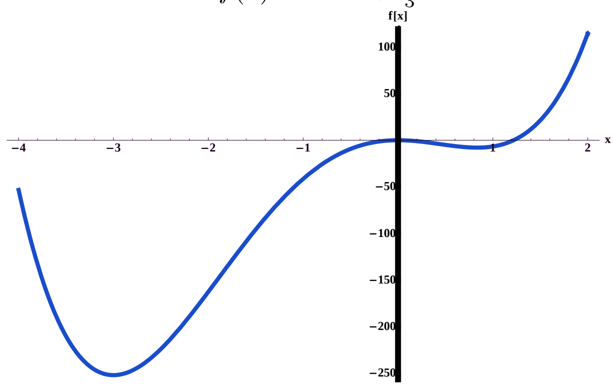
$$\det[D] = -1$$

$$\det[D] = \begin{vmatrix} 1 & -4 & 2 \\ -3 & 13 & -7 \\ 4 & -4 & -5 \end{vmatrix}$$
$$= 1 \times \begin{vmatrix} 13 & -7 \\ -4 & -5 \end{vmatrix} - (-4) \times \begin{vmatrix} -3 & -7 \\ 4 & -5 \end{vmatrix} + 2 \times \begin{vmatrix} -3 & 13 \\ 4 & -4 \end{vmatrix}$$
$$= -93 + 172 - 80 = -1$$

Problem 8. For each of the following problems, find the critical points. For each critical point state whether the function is at a relative maximum, relative minimum, or otherwise. Check to see if there are potential points of inflection **at points other than** critical points.

a. $f(x) \ = 6x^4 + \frac{52}{3}x^3 - 30x^2$.

FIGURE 1. $f(x) \ = \ 6x^4 + \frac{52}{3}x^3 - 30x^2$



The derivatives of $f(x) = 6x^4 + \frac{52}{3}x^3 - 30x^2$ are given by

$$f'(x) = 24x^3 + 52x^2 - 60x \tag{43}$$

$$f''(x) = 72x^2 + 104x - 60 \tag{44}$$

$$f^{(3)}(x) = 144x + 104 \tag{45}$$

Set the first derivative, equation (43), to be zero.

$$\begin{aligned} f'(x) = 24x^3 + 52x^2 - 60x &= 0 \\ \Rightarrow \qquad x(6x^2 + 13x - 15) &= 0 \\ \Rightarrow \qquad x(x+3)(6x-5) &= 0 \\ \Rightarrow \qquad x = 0 \quad \text{or} \quad x = -3 \quad &\text{or} \quad x = 5/6 \end{aligned}$$

Check the second derivative, equation (44), for $x = 0$, $x = -3$, and $x = 5/6$ respectively.

$$f''(0) = 72 \times 0^2 + 104 \times 0 - 60 = -60 < 0$$

$$f''(-3) = 72 \times (-3)^2 + 104 \times (-3) - 60 = 276 > 0$$

$$f''(5/6) = 72 \times (5/6)^2 + 104 \times (5/6) - 60 = 230/3 > 0$$

So $x = 0$ is a local maximum point; $x = -3$ and $x = 5/6$ are local maximum points.

Set the second derivative, equation (44), to be zero.

$$\begin{aligned} f''(x) = 72x^2 + 104x - 60 &= 0 \\ \Rightarrow \qquad 18x^2 + 26x - 15 &= 0 \\ \Rightarrow \qquad x &= \frac{-26 \pm \sqrt{26^2 - 4 \times 18 \times (-15)}}{36} \\ \Rightarrow \qquad x &= \frac{-13 \pm \sqrt{439}}{18} \end{aligned}$$

Since $f^{(3)}(x) = 144x + 104$ is zero only when $x = -104/144 = 13/18$, $x = \dfrac{-13 \pm \sqrt{439}}{18}$ are inflection points.

b. $f(x) = 4e^{-3x}\left(\frac{28}{9} - \frac{17x}{3} - 2x^2\right)$

FIGURE 2. $f(x) = 4e^{-3x}\left(\frac{28}{9} - \frac{17x}{3} - 2x^2\right)$

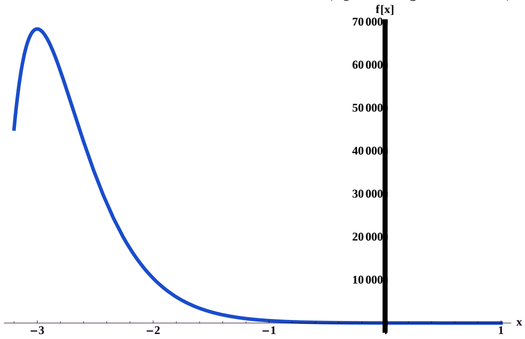
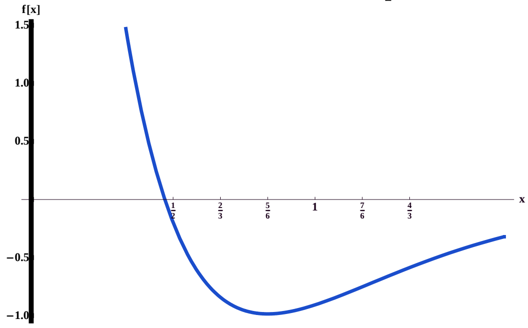


FIGURE 3. A Close-up



Hints: The second derivative of the function will simplify to $-72e^{-3x}x^2 - 108e^{-3x}x + 232e^{-3x}$. The real points of inflection (you still need to find them) are $\frac{1}{12}\left(-9 \pm \sqrt{545}\right)$. $9 \times 545 = 4905$.

The derivatives of $f(x) = 4e^{-3x}\left(\frac{28}{9} - \frac{17x}{3} - 2x^2\right)$ are given by

$$\begin{aligned}
f'(x) &= 4e^{-3x}\cdot(-3)\left(\frac{28}{9} - \frac{17x}{3} - 2x^2\right) + 4e^{-3x}\left(\frac{-17}{3} - 4x\right) \\
&= e^{-3x}\left(-60 + 52x + 24x^2\right) \qquad (46) \\
f''(x) &= e^{-3x}\cdot(-3)\left(-60 + 52x + 24x^2\right) + e^{-3x}(52 + 48x) \\
&= e^{-3x}\left(232 - 108x - 72x^2\right) \qquad (47) \\
f^{(3)}(x) &= e^{-3x}\cdot(-3)\left(232 - 108x - 72x^2\right) + e^{-3x}(-108 - 144x) \\
&= e^{-3x}\left(-804 + 180x + 216x^2\right) \\
&= 12e^{-3x}\left(-67 + 15x + 18x^2\right) \qquad (48)
\end{aligned}$$

Set the first derivative, equation (46), to zero.

$$\begin{aligned}
f'(x) = e^{-3x}\left(-60 + 52x + 24x^2\right) &= 0 \\
\Rightarrow \quad -60 + 52x + 24x^2 &= 0 \\
\Rightarrow \quad 4(-15 + 13x + 6x^2) &= 0 \\
\Rightarrow \quad -4(x+3)(6x-5) &= 0 \\
\Rightarrow \quad x = -3 \quad \text{or} \quad & x = 5/6
\end{aligned}$$

Check the second derivative, equation (47), for $x = -3$ and $x = 5/6$ respectively.

$$\begin{aligned}
f''(-3) &= e^{-3\times(-3)}\left(232 - 108\times(-3) - 72\times(-3)^2\right) \\
&= e^9 \times (-92) < 0 \\
f''(5/6) &= e^{-3\times(5/6)}\left(232 - 108\times(5/6) - 72\times(5/6)^2\right) \\
&= e^{-5/2}\times 92 > 0
\end{aligned}$$

So $x = -3$ is a local maximum point; $x = 5/6$ is a local minimum point.

Set the second derivative, equation (47), to be zero.

$$\begin{aligned}
f''(x) = e^{-3x}\left(232 - 108x - 72x^2\right) &= 0 \\
\Rightarrow \quad 232 - 108x - 72x^2 &= 0 \\
\Rightarrow \quad 58 - 27x - 18x^2 &= 0 \\
\Rightarrow \quad x &= \frac{27 \pm \sqrt{27^2 - 4 \times (-18) \times 58}}{-36} \\
\Rightarrow \quad x &= \frac{9 \pm \sqrt{545}}{-12}
\end{aligned}$$

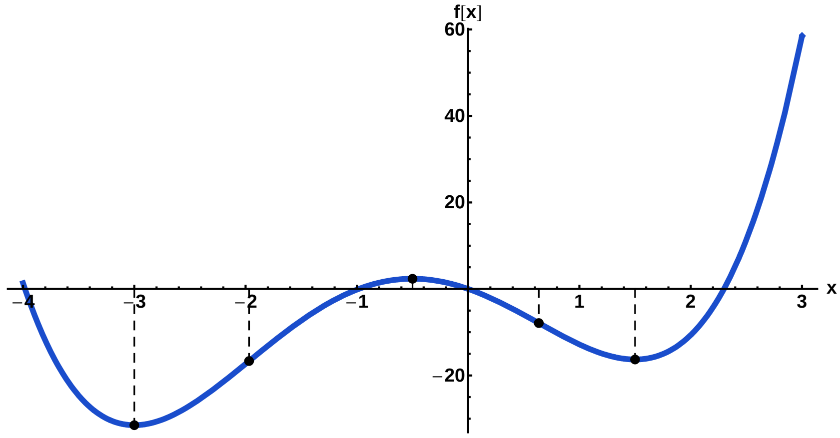
Also set third derivative, equation (48), to be zero.

$$\begin{aligned}
f^{(3)}(x) = 12e^{-3x}\left(-67 + 15x + 18x^2\right) &= 0 \\
\Rightarrow \quad -67 + 15x + 18x^2 &= 0 \\
\Rightarrow \quad x &= \frac{-15 \pm \sqrt{15^2 - 4 \times 18 \times (-67)}}{36} \\
\Rightarrow \quad x &= \frac{-5 \pm \sqrt{761}}{12}
\end{aligned}$$

As a result, when $x = \frac{9 \pm \sqrt{545}}{-12} \neq \frac{-5 \pm \sqrt{761}}{12}$, the third derivatives are not equal to zero. Thus, $x = \frac{9 \pm \sqrt{545}}{-12}$ are two inflection points.

c. $f(x) \;=\; x^4 + \frac{8x^3}{3} - \frac{15x^2}{2} - 9x$

Figure 4. $f(x) \;=\; x^4 + \frac{8x^3}{3} - \frac{15x^2}{2} - 9x$



The inflection points are $\frac{1}{6}\left(-4 \pm \sqrt{61}\right)$.

The derivatives of $f(x) = x^4 + \frac{8x^3}{3} - \frac{15x^2}{2} - 9x$ are given by

$$f'(x) = 4x^3 + 8x^2 - 15x - 9 \tag{49}$$

$$f''(x) = 12x^2 + 16x - 15 \tag{50}$$

$$f^{(3)}(x) = 24x + 16 \tag{51}$$

Set the first derivative, equation (49), to be zero.

$$f'(x) = 4x^3 + 8x^2 - 15x - 9 = 0 \tag{52}$$

By guess, $x = -3$ is a solution of equation $f'(x) = 0$. Then we try to factorize $f'(x) = 4x^3 + 8x^2 - 15x - 9 = 0$ by $(x + 3)$.

$$\begin{array}{r|l} & \quad 4x^2 \;\; - 4x - 3 \\ \hline x+3 & \;\; 4x^3 \;\; + 8x^2 - 15x - 9 \\ & - 4x^3 - 12x^2 \\ \hline & \qquad - 4x^2 - 15x \\ & \qquad\quad 4x^2 + 12x \\ \hline & \qquad\qquad - 3x - 9 \\ & \qquad\qquad\quad 3x + 9 \\ \hline & \qquad\qquad\qquad 0 \end{array}$$

Then, equation (52) can be easily solved.

$$\begin{aligned} f'(x) = 4x^3 + 8x^2 - 15x - 9 &= 0 \\ \Rightarrow \qquad (x+3)(4x^2 - 4x - 3) &= 0 \\ \Rightarrow \qquad (x+3)(2x+1)(2x-3) &= 0 \\ \Rightarrow \quad x = -3 \quad \text{or} \quad x = -1/2 \quad &\text{or} \quad x = 3/2 \end{aligned}$$

Check the second derivative, equation (50), for $x = -3$, $x = -1/2$, and $x = 3/2$.

$$f''(-3) = 12 \times (-3)^2 + 16 \times (-3) - 15 = 45 > 0$$

$$f''(-1/2) = 12 \times (-1/2)^2 + 16 \times (-1/2) - 15 = -20 < 0$$

$$f''(3/2) = 12 \times (3/2)^2 + 16 \times (3/2) - 15 = 36 > 0$$

So $x = -3$ and $x = 3/2$ are local minimum points; $x = -1/2$ is a local minimum point.

Set the second derivative, equation (50), to be zero.

$$\begin{aligned} f''(x) = 12x^2 + 16x - 15 &= 0 \\ \Rightarrow \qquad x &= \frac{-16 \pm \sqrt{16^2 - 4 \times 12 \times (-15)}}{24} \\ \Rightarrow \qquad x &= \frac{-4 \pm \sqrt{61}}{6} \end{aligned}$$

Since $f^{(3)}(x) = 24x + 16$ is zero only when $x = -16/24 = -2/3$, $x = \dfrac{-4 \pm \sqrt{61}}{6}$ are inflection points.