ECONOMICS 207
SPRING 2008
PROBLEM SET 11
KEY

Problem 1. Consider the following matrix and vector.

$$P = \begin{bmatrix} 1 & 5 \\ 1 & 6 \end{bmatrix}, \quad p = \begin{bmatrix} 7 \\ 9 \end{bmatrix},$$

a. Use elementary row operations to find both the inverse of P and solve the equation Px=p in one set of operations.

First augment matrix P with an 2×2 identity matrix and matrix p. That is, let

$$\tilde{A} = \begin{pmatrix} P & I_{2\times 2} & p \end{pmatrix} = \begin{pmatrix} 1 & 5 & 1 & 0 & 7 \\ 1 & 6 & 0 & 1 & 9 \end{pmatrix}$$

Then use elementary row operations on matrix $\tilde{A}$ to create an identity matrix on the left side. To begin with, based on matrix $\tilde{A}$, subtract the first row from the second row.

$$\begin{pmatrix} 1 & 5 & 1 & 0 & 7 \\ 1-1 & 6-5 & 0-1 & 1 & 9-7 \end{pmatrix} = \begin{pmatrix} 1 & 5 & 1 & 0 & 7 \\ 0 & 1 & -1 & 1 & 2 \end{pmatrix} \tag{1}$$

Based on the matrix on the right side of equation (1), add the second row multiplied by -5 to the first row.

$$\begin{pmatrix} 1 & 5+1\times(-5) & 1+(-1)\times(-5) & 0+1\times(-5) & 7+2\times(-5) \\ 0 & 1 & -1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 6 & -5 & -3 \\ 0 & 1 & -1 & 1 & 2 \end{pmatrix} \tag{2}$$

So $P^{-1} = \begin{pmatrix} 6 & -5 \\ -1 & 1 \end{pmatrix}$ and the solution for $P\mathbf{x} = P$ is $\mathbf{x} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$.

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b. Find the determinant of the matrix P.

$$P = \begin{bmatrix} 1 & 5 \\ 1 & 6 \end{bmatrix}, \quad p = \begin{bmatrix} 7 \\ 9 \end{bmatrix},$$

$$\det[P] = 1 \times 6 - 1 \times 5 = 1$$

c. Find the inverse of the matrix P using the cofactor/adjoint method.

The adjoint matrix of P is given by

$$\text{adj}(P) = \begin{pmatrix} 6 & -1 \\ -5 & 1 \end{pmatrix}' = \begin{pmatrix} 6 & -5 \\ -1 & 1 \end{pmatrix}$$

Therefore the inverse of matrix P is given by

$$P^{-1} = \frac{\text{adj}(P)}{\det[P]} = \frac{\text{adj}(P)}{1} = \text{adj}(P) = \begin{pmatrix} 6 & -5 \\ -1 & 1 \end{pmatrix}$$

d. Solve the equation Px=p using the inverse you found in part 1c

Use the inverse of P, the solution is given by

$$\mathbf{x} = P^{-1}p = \begin{pmatrix} 6 & -5 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ 9 \end{pmatrix} = \begin{pmatrix} 6 \times 7 - 5 \times 9 \\ (-1) \times 7 + 1 \times 9 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

e. Solve the equation Px=p using Cramer's rule.

$$P = \begin{bmatrix} 1 & 5 \\ 1 & 6 \end{bmatrix}, \quad p = \begin{bmatrix} 7 \\ 9 \end{bmatrix},$$

By Cramer's rule, the solution is given by

$$\mathbf{x} = \frac{\begin{pmatrix} \begin{vmatrix} 7 & 5 \\ 9 & 6 \end{vmatrix} \\ \\ \begin{vmatrix} 1 & 7 \\ 1 & 9 \end{vmatrix} \end{pmatrix}}{\det[P]}$$

$$= \begin{pmatrix} \begin{vmatrix} 7 & 5 \\ 9 & 6 \end{vmatrix} \\ \\ \begin{vmatrix} 1 & 7 \\ 1 & 9 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

Problem 2. Consider the following matrix and vector.

$$Q = \begin{bmatrix} 2 & 2 \\ 5 & 4 \end{bmatrix}, \quad q = \begin{bmatrix} 4 \\ 12 \end{bmatrix},$$

a. Use elementary row operations to find both the inverse of Q and solve the equation Qx=q in one set of operations.

First augment matrix Q with an 2×2 identity matrix and matrix q. That is, let

$$\tilde{A} = \begin{pmatrix} Q & I_{2\times 2} & q \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 & 0 & 4 \\ 5 & 4 & 0 & 1 & 12 \end{pmatrix}$$

Then use elementary row operations on matrix $\tilde{A}$ to create an identity matrix on the left side.

To begin with, based on matrix $\tilde{A}$, subtract the first row multiplied by $5/2$ from the second row.

$$\begin{pmatrix} 2 & 2 & 1 & 0 & 4 \\ 5-2\times(5/2) & 4-2\times(5/2) & 0-1\times(5/2) & 1 & 12-4\times(5/2) \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 & 0 & 4 \\ 0 & -1 & -5/2 & 1 & 2 \end{pmatrix} \tag{3}$$

Based on the matrix on the right side of equation (3), add the second row multiplied by 2 to the first row.

$$\begin{pmatrix} 2 & 2+(-1)\times 2 & 1+(-5/2)\times 2 & 0+1\times 2 & 4+2\times 2 \\ 0 & -1 & -5/2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -4 & 2 & 8 \\ 0 & -1 & -5/2 & 1 & 2 \end{pmatrix} \tag{4}$$

Based on the matrix on the right side of equation (4) divide the first row by 2; divide the second row by -1;

$$\begin{pmatrix} 2/2 & 0 & -4/2 & 2/2 & 8/2 \\ 0 & -1/(-1) & -5/2/(-1) & 1/(-1) & 2/(-1) \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 & 1 & 4 \\ 0 & 1 & 5/2 & -1 & -2 \end{pmatrix} \tag{5}$$

So $Q^{-1} = \begin{pmatrix} -2 & 1 \\ 5/2 & -1 \end{pmatrix}$ and the solution for $Q\mathbf{x} = q$ is $\mathbf{x} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

b. Find the determinant of the matrix Q.

$$Q = \begin{bmatrix} 2 & 2 \\ 5 & 4 \end{bmatrix}, \quad q = \begin{bmatrix} 4 \\ 12 \end{bmatrix},$$

$$\det[Q] = 2 \times 4 - 2 \times 5 = -2$$

c. Find the inverse of the matrix Q using the cofactor/adjoint method.

The adjoint matrix of Q is given by

$$\text{adj}(P) = \begin{pmatrix} 4 & -5 \\ -2 & 2 \end{pmatrix}' = \begin{pmatrix} 4 & -2 \\ -5 & 2 \end{pmatrix}$$

Therefore the inverse of matrix Q is given by

$$Q^{-1} = \frac{\text{adj}(Q)}{\det[Q]} = \frac{\text{adj}(Q)}{-2} = \begin{pmatrix} 4/(-2) & -2/(-2) \\ -5/(-2) & 2/(-2) \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 5/2 & -1 \end{pmatrix}$$

d. Solve the equation Qx=q using the inverse you found in part 2c

Use the inverse of Q, the solution is given by

$$\mathbf{x} = Q^{-1}q = \begin{pmatrix} -2 & 1 \\ 5/2 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 12 \end{pmatrix} = \begin{pmatrix} -2 \times 4 + 1 \times 12 \\ (5/2) \times 4 + (-1) \times 12 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

e. Solve the equation Qx=q using Cramer's rule.

$$Q = \begin{bmatrix} 2 & 2 \\ 5 & 4 \end{bmatrix}, \quad q = \begin{bmatrix} 4 \\ 12 \end{bmatrix},$$

By Cramer's rule, the solution is given by

$$\begin{aligned} \mathbf{x} &= \frac{\begin{pmatrix} \begin{vmatrix} 4 & 2 \\ 12 & 4 \end{vmatrix} \\ \\ \begin{vmatrix} 2 & 4 \\ 5 & 12 \end{vmatrix} \end{pmatrix}}{\det[P]} \\ &= \frac{\begin{pmatrix} -8 \\ 4 \end{pmatrix}}{-2} \\ &= \begin{pmatrix} 4 \\ -2 \end{pmatrix} \end{aligned}$$

Problem 3. Consider the following matrix and vector.

$$G = \begin{bmatrix} -3 & -3 & -4 \\ 0 & -1 & -2 \\ 2 & 1 & 1 \end{bmatrix}, \quad g = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

a. Use elementary row operations to find the inverse of G and solve the equation Gx=g in one set of operations.

Augment matrix G with an 3×3 identity matrix and matrix g. That is, let

$$\begin{aligned} \tilde{A} &= \begin{pmatrix} G & I_{3\times 3} & g \end{pmatrix} \\ &= \begin{pmatrix} -3 & -3 & -4 & 1 & 0 & 0 & 1 \\ 0 & -1 & -2 & 0 & 1 & 0 & 4 \\ 2 & 1 & 1 & 0 & 0 & 1 & 3 \end{pmatrix} \end{aligned}$$

Then use elementary row operations on $\tilde{A}$ to create an identity matrix on the left side. To begin with, based on $\tilde{A}$, add the first row multiplied by 2/3 to the third row.

$$\begin{pmatrix} -3 & -3 & -4 & 1 & 0 & 0 & 1 \\ 0 & -1 & -2 & 0 & 1 & 0 & 4 \\ 2-3\times(2/3) & 1-3\times(2/3) & 1-4\times(2/3) & 0+1\times(2/3) & 0 & 1 & 3+1\times(2/3) \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -3 & -4 & 1 & 0 & 0 & 1 \\ 0 & -1 & -2 & 0 & 1 & 0 & 4 \\ 0 & -1 & -5/3 & 2/3 & 0 & 1 & 11/3 \end{pmatrix} \tag{6}$$

Based on the matrix on the right side of equation (6), subtract the second row from the third row.

$$\begin{pmatrix} -3 & -3 & -4 & 1 & 0 & 0 & 1 \\ 0 & -1 & -2 & 0 & 1 & 0 & 4 \\ 0 & -1-(-1) & -5/3-(-2) & 2/3 & 0-1 & 1 & 11/3-4 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -3 & -4 & 1 & 0 & 0 & 1 \\ 0 & -1 & -2 & 0 & 1 & 0 & 4 \\ 0 & 0 & 1/3 & 2/3 & -1 & 1 & -1/3 \end{pmatrix} \tag{7}$$

Based on the matrix on the right side of equation (7), multiply the third row by 3.

$$\begin{pmatrix} -3 & -3 & -4 & 1 & 0 & 0 & 1 \\ 0 & -1 & -2 & 0 & 1 & 0 & 4 \\ 0 & 0 & 1/3\times 3 & 2/3\times 3 & -1\times 3 & 1\times 3 & -1/3\times 3 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -3 & -4 & 1 & 0 & 0 & 1 \\ 0 & -1 & -2 & 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 & -3 & 3 & -1 \end{pmatrix} \tag{8}$$

Based on the matrix on the right side of equation (8) add the third row multiplied by 2 to the second row; add the third row multiplied by 4 to the first row;

$$\begin{pmatrix} -3 & -3 & -4+1\times 4 & 1+2\times 4 & 0+(-3)\times 4 & 0+3\times 4 & 1+(-1)\times 4 \\ 0 & -1 & -2+1\times 2 & 0+2\times 2 & 1-3\times 2 & 0+3\times 2 & 4-1\times 2 \\ 0 & 0 & 1 & 2 & -3 & 3 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -3 & 0 & 9 & -12 & 12 & -3 \\ 0 & -1 & 0 & 4 & -5 & 6 & 2 \\ 0 & 0 & 1 & 2 & -3 & 3 & -1 \end{pmatrix} \tag{9}$$

Based on the matrix on the right side of equation (9), divide the second row by -1.

$$\begin{pmatrix} -3 & -3 & 0 & 9 & -12 & 12 & -3 \\ 0 & -1/(-1) & 0 & 4/(-1) & -5/(-1) & 6/(-1) & 2/(-1) \\ 0 & 0 & 1 & 2 & -3 & 3 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -3 & 0 & 9 & -12 & 12 & -3 \\ 0 & 1 & 0 & -4 & 5 & -6 & -2 \\ 0 & 0 & 1 & 2 & -3 & 3 & -1 \end{pmatrix} \tag{10}$$

Based on the matrix on the right side of equation (10), add the second row multiplied by 3 to the first row.

$$\begin{pmatrix} -3 & -3+1\times 3 & 0 & 9+(-4)\times 3 & -12+5\times 3 & 12+(-6)\times 3 & -3+(-2)\times 3 \\ 0 & 1 & 0 & -4 & 5 & -6 & -2 \\ 0 & 0 & 1 & 2 & -3 & 3 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 0 & 0 & -3 & 3 & -6 & -9 \\ 0 & 1 & 0 & -4 & 5 & -6 & -2 \\ 0 & 0 & 1 & 2 & -3 & 3 & -1 \end{pmatrix} \tag{11}$$

Based on the matrix on the right side of equation (11), divide the first row by -3.

$$\begin{pmatrix} -3/(-3) & 0 & 0 & -3/(-3) & 3/(-3) & -6/(-3) & -9/(-3) \\ 0 & 1 & 0 & -4 & 5 & -6 & -2 \\ 0 & 0 & 1 & 2 & -3 & 3 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 1 & -1 & 2 & 3 \\ 0 & 1 & 0 & -4 & 5 & -6 & -2 \\ 0 & 0 & 1 & 2 & -3 & 3 & -1 \end{pmatrix} \tag{12}$$

So $G^{-1} = \begin{pmatrix} 1 & -1 & 2 \\ -4 & 5 & -6 \\ 2 & -3 & 3 \end{pmatrix}$ and the solution for $G\mathbf{x} = g$ is $\mathbf{x} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$.

b. Find the determinant of the matrix G.

$$G = \begin{bmatrix} -3 & -3 & -4 \\ 0 & -1 & -2 \\ 2 & 1 & 1 \end{bmatrix}, \quad g = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

Find the determinant of the matrix G by expansion of $|G|$ in terms of the first column.

$$\det[G] = -3 \times \begin{vmatrix} -1 & -2 \\ 1 & 1 \end{vmatrix} + 2 \times \begin{vmatrix} -3 & -4 \\ -1 & -2 \end{vmatrix}$$

$$= -3 + 4 = 1$$

c. Find the inverse of the matrix G using the cofactor/adjoint method.

$$G = \begin{bmatrix} -3 & -3 & -4 \\ 0 & -1 & -2 \\ 2 & 1 & 1 \end{bmatrix}, \quad g = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

The adjoint matrix of G is given by

$$\operatorname{adj}(G) = \begin{pmatrix} \begin{vmatrix} -1 & -2 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & -2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 0 & -1 \\ 2 & 1 \end{vmatrix} \\ -\begin{vmatrix} -3 & -4 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} -3 & -4 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} -3 & -3 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} -3 & -4 \\ -1 & -2 \end{vmatrix} & -\begin{vmatrix} -3 & -4 \\ 0 & -2 \end{vmatrix} & \begin{vmatrix} -3 & -3 \\ 0 & -1 \end{vmatrix} \end{pmatrix}^T$$

$$= \begin{pmatrix} 1 & -4 & 2 \\ -1 & 5 & -3 \\ 2 & -6 & 3 \end{pmatrix}^T$$

$$= \begin{pmatrix} 1 & -1 & 2 \\ -4 & 5 & -6 \\ 2 & -3 & 3 \end{pmatrix}$$

Then the inverse of G is given by

$$G^{-1} = \frac{\operatorname{adj}(G)}{\det[G]}$$

$$= \frac{\operatorname{adj}(G)}{1} = \operatorname{adj}(G)$$

$$= \begin{pmatrix} 1 & -1 & 2 \\ -4 & 5 & -6 \\ 2 & -3 & 3 \end{pmatrix}$$

d. Solve the equation Gx=g using the inverse you found in part 3c

$$G = \begin{bmatrix} -3 & -3 & -4 \\ 0 & -1 & -2 \\ 2 & 1 & 1 \end{bmatrix}, \quad g = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

Using the inverse of G, the solution is given by

$$\begin{aligned} \mathbf{x} &= G^{-1}g \\ &= \begin{pmatrix} 1 & -1 & 2 \\ -4 & 5 & -6 \\ 2 & -3 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 \times 1 + (-1) \times 4 + 2 \times 3 \\ -4 \times 1 + 5 \times 4 + (-6) \times 3 \\ 2 \times 1 + (-3) \times 4 + 3 \times 3 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} \end{aligned}$$

e. Solve the equation Gx=g using Cramer's rule.

By Cramer's rule, the solution is given by

$$\mathbf{x} = \frac{\begin{pmatrix} \begin{vmatrix} 1 & -3 & -4 \\ 4 & -1 & -2 \\ 3 & 1 & 1 \end{vmatrix} \\ \begin{vmatrix} -3 & 1 & -4 \\ 0 & 4 & -2 \\ 2 & 3 & 1 \end{vmatrix} \\ \begin{vmatrix} -3 & -3 & 1 \\ 0 & -1 & 4 \\ 2 & 1 & 3 \end{vmatrix} \end{pmatrix}}{\det[G]}$$

$$= \begin{pmatrix} 1 \times \begin{vmatrix} -1 & -2 \\ 1 & 1 \end{vmatrix} - (-3) \times \begin{vmatrix} 4 & -2 \\ 3 & 1 \end{vmatrix} - 4 \times \begin{vmatrix} 4 & -1 \\ 3 & 1 \end{vmatrix} \\ -3 \times \begin{vmatrix} 4 & -2 \\ 3 & 1 \end{vmatrix} + 2 \times \begin{vmatrix} 1 & -4 \\ 4 & -2 \end{vmatrix} \\ -3 \times \begin{vmatrix} -1 & 4 \\ 1 & 3 \end{vmatrix} + 2 \times \begin{vmatrix} -3 & 1 \\ -1 & 4 \end{vmatrix} \end{pmatrix}$$

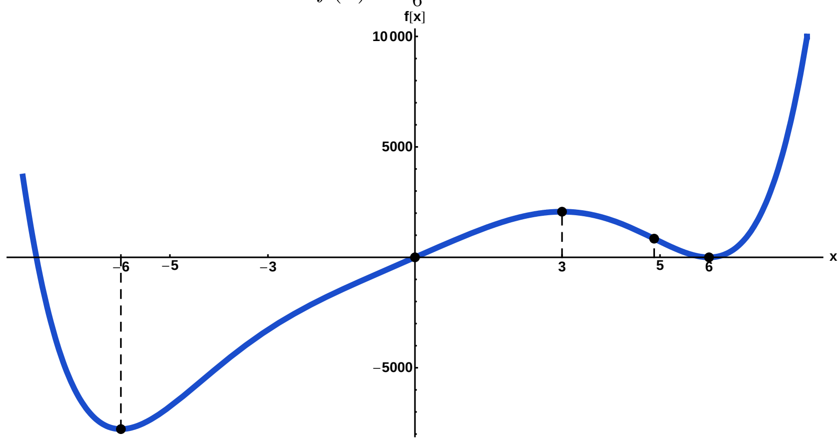
$$= \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$$

Problem 4. For each of the following problems, find the critical points. For each critical point state whether the function is at a relative maximum, relative minimum, or otherwise. Check to see if there are potential points of inflection **at points other than** critical points.

a. $f(x) = \frac{x^6}{6} - 9x^4 - 9x^3 + 972x.$

Hint: You might consider dividing the polynomial in the first order condition by appropriate polynomials one after another or maybe try dividing by $(x^3 - 27)$. The only real inflection point is zero. Show that this is true.

FIGURE 1. $f(x) = \frac{x^6}{6} - 9x^4 - 9x^3 + 972x$



The derivatives of $f(x) = \frac{x^6}{6} - 9x^4 - 9x^3 + 972x$ are given by

$$f'(x) = x^5 - 36x^3 - 27x^2 + 972 \tag{13}$$

$$f''(x) = 5x^4 - 108x^2 - 54x \tag{14}$$

$$f^{(3)}(x) = 20x^3 - 216x - 54 \tag{15}$$

Set the first derivative, equation (13), to zero.

$$f'(x) = x^5 - 36x^3 - 27x^2 + 972 \tag{16}$$

By hint, we try to factorize $f'(x) = x^5 - 36x^3 - 27x^2 + 972$ by $(x^3 - 27)$.

$$\begin{array}{r|l} & x^2 \quad - 36 \\ \hline x^3 - 27 & x^5 - 36x^3 - 27x^2 + 972 \\ & -x^5 \qquad\qquad + 27x^2 \\ \hline & -36x^3 \qquad\qquad + 972 \\ & 36x^3 \qquad\qquad - 972 \\ \hline & 0 \end{array}$$

The we can solve equation (13).

$$\begin{aligned} f'(x) = x^5 - 36x^3 - 27x^2 + 972 &= 0 \\ \Rightarrow \qquad (x^3 - 27)(x^2 - 36) &= 0 \\ \Rightarrow \qquad x = 3 \qquad &\text{or} \qquad x = \pm 6 \end{aligned}$$

Check the second derivatives, equation (14), for $x = 3$ and $x = \pm 6$.

$$f''(3) = 5 \times 3^4 - 108 \times 3^2 - 54 \times 3 = -729 < 0$$

$$f''(6) = 5 \times 6^4 - 108 \times 6^2 - 54 \times 6 = 2268 > 0$$

$$f''(-6) = 5 \times (-6)^4 - 108 \times (-6)^2 - 54 \times (-6) = 2916 > 0$$

So $x = 3$ is a local maximum point; $x = \pm 6$ are local minimum points.

Set the second derivative, equation (14), to be zero.

$$f''(x) = 5x^4 - 108x^2 - 54x = x(5x^3 - 108x - 54) = 0$$

It is easy to find that $x = 0$ is a solution of equation $f''(x) = 0$. And when $x = 0$, $f^{(3)}(0) = -54 \neq 0$. Thus, $x = 0$ is an inflection point.

Let $g(x) = 5x^3 - 108x - 54$. So the solution of $f''(x) = xg(x) = 0$ is $x = 0$ and solutions of equation $g(x) = 0$. Then we are trying to analyze equation $g(x) = 0$.

First, the first derivative of $g(x)$ is given by

$$g'(x) = 15x^2 - 108$$

And then the solutions for $g'(x) = 0$ is $x = \pm\sqrt{\frac{108}{15}} = \pm\frac{6\sqrt{5}}{5}$. At $x = \pm\frac{6\sqrt{5}}{5}$, i.e., $x^2 = 36/5$,

$$\begin{aligned} g(x)\big|_{x=\pm\frac{6\sqrt{5}}{5}} &= x(5x^2 - 108) - 54\big|_{x=\pm\frac{6\sqrt{5}}{5}} \\ &= -72x - 54\big|_{x=\pm\frac{6\sqrt{5}}{5}} \end{aligned}$$

Respectively,

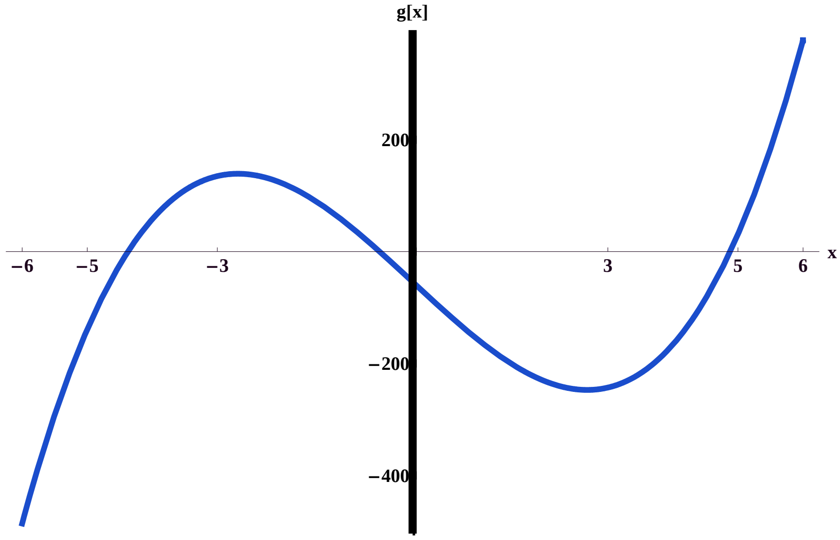
$$g\left(\frac{6\sqrt{5}}{5}\right) = \frac{-432\sqrt{5}}{5} - 54 < 0$$

$$g\left(-\frac{6\sqrt{5}}{5}\right) = \frac{432\sqrt{5}}{5} - 54 > 0$$

Then we can conclude that $g(x) = 0$ has three real solutions. Also this could be indicated by the graph of $g(x)$, see figure 2.

So besides inflection point $x = 0$, $f(x)$ has other three potential inflection points.

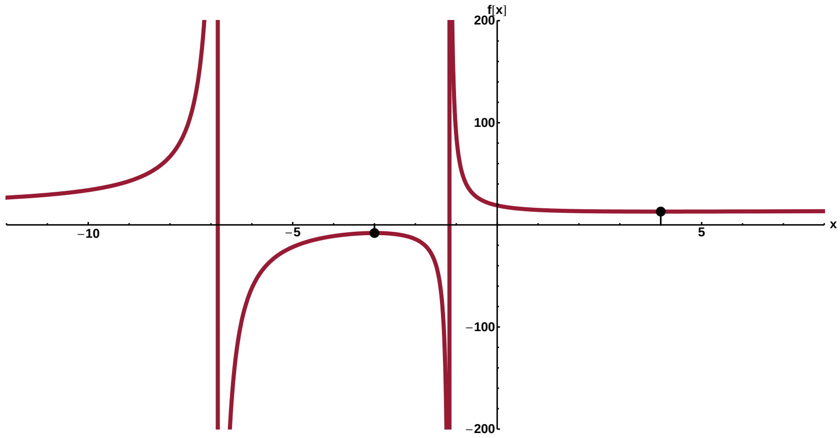
FIGURE 2. $g(x) = 5x^3 - 108x - 54$



b. $f(x) \;=\; \dfrac{4x^2+20x+38}{\frac{x^2}{4}+2x+2}$

You need not find the points of inflection for this problem. Hint: The first derivative simplifies to $\frac{48\left(x^2-x-12\right)}{(x^2+8x+8)^2}$. You should find the second derivative of f(x) but here is the answer: $-\frac{48\left(2x^3-3x^2-7x-184\right)}{(x^2+8x+8)^3}$.

FIGURE 3. $f(x) \;=\; \dfrac{4x^2+20x+38}{\frac{x^2}{4}+2x+2}$



Simplify $f(x)$ first.

$$f(x) = \frac{4x^2+20x+38}{\frac{x^2}{4}+2x+2} = \frac{4x^2+32x+32-12x+6}{\frac{x^2}{4}+2x+2}$$

$$= 16 + \frac{-12x+6}{\frac{x^2}{4}+2x+2} = 16 + (-12x+6)\left(\frac{x^2}{4}+2x+2\right)^{-1}$$

The derivatives of $f(x)$ are given by

$$\begin{aligned}
f'(x) &= -12\left(\frac{x^2}{4}+2x+2\right)^{-1} + (-12x+6)\cdot(-1)\left(\frac{x^2}{4}+2x+2\right)^{-2}(x/2+2) \\
&= \left(\frac{x^2}{4}+2x+2\right)^{-2}\left(-12\left(\frac{x^2}{4}+2x+2\right) + \left(6x^2+21x-12\right)\right) \\
&= \left(\frac{x^2}{4}+2x+2\right)^{-2}\left(3x^2-3x-36\right) \\
&= 48\left(x^2+8x+8\right)^{-2}\left(x^2-x-12\right) \qquad (17) \\
f''(x) &= 48\left[(-2)\left(x^2+8x+8\right)^{-3}\cdot(2x+8)\left(x^2-x-12\right) + \left(x^2+8x+8\right)^{-2}(2x-1)\right] \\
&= 48\left(x^2+8x+8\right)^{-3}\left[-2(2x+8)(x^2-x-12) + (x^2+8x+8)(2x-1)\right] \\
&= -48\left(x^2+8x+8\right)^{-3}\left(2x^3-3x^2-72x-184\right) \qquad (18)
\end{aligned}$$

Set the first derivative, equation (17), to zero.

$$\begin{aligned}
f'(x) = 48\left(x^2+8x+8\right)^{-2}\left(x^2-x-12\right) &= 0 \\
\Rightarrow \qquad x^2-x-12 &= 0 \\
\Rightarrow \qquad (x-4)(x+3) &= 0 \\
\Rightarrow \qquad x=4 \quad &\text{or} \quad 0 \qquad x=-3
\end{aligned}$$

Check the second derivative, equation (18), for $x=-3$ and $x=4$ respectively.

$$\begin{aligned}
f''(-3) &= -48\left((-3)^2+8\times(-3)+8\right)^{-3}\left(2\times(-3)^3-3\times(-3)^2-72\times(-3)-184\right) \\
&= -\frac{48}{7} < 0 \\
f''(4) &= -48\left(4^2+8\times 4+8\right)^{-3}\left(2\times 4^3-3\times 4^2-72\times 4-184\right) \\
&= \frac{3}{28} > 0
\end{aligned}$$

So $x=-3$ is a local maximum point; $x=4$ is a local minimum point.

Problem 5. Solve the following system of equations.

$$1944x_1^{-5/6}x_2^{1/3} - 32 = 0$$

$$3888x_1^{1/6}x_2^{-2/3} - 729 = 0$$

Hint: $x_1 = 729$.

Rearrange the first equation.

$$\begin{aligned} 1944x_1^{-5/6}x_2^{1/3} - 32 &= 0 \\ \Rightarrow \quad x_1^{-5/6}x_2^{1/3} &= 32/1944 = 4/243 \\ \Rightarrow \quad x_2^{1/3} &= \tfrac{4}{243}x_1^{5/6} \\ \Rightarrow \quad x_2^{-2/3} &= \tfrac{243^2}{4^2}x_1^{-10/6} \end{aligned}$$

Substitute $x_2^{-2/3} = \frac{243^2}{4^2}x_1^{-10/6}$ into the second equation.

$$\begin{aligned} 3888x_1^{1/6}x_2^{-2/3} - 729 &= 0 \\ \Rightarrow \quad 3888x_1^{1/6}\left(\tfrac{243^2}{4^2}x_1^{-10/6}\right) - 729 &= 0 \\ \Rightarrow \quad x_1^{-3/2} &= \tfrac{729\times 4^2}{3888\times 243^2} = 3^{-9} \\ \Rightarrow \quad x_1 &= \left(3^{-9}\right)^{-2/3} = 3^6 = 729 \end{aligned}$$

Substitute $x_1 = 729$ into $x_2^{1/3} = \frac{4}{243}x_1^{5/6}$.

$$\begin{aligned} x_2^{1/3} &= \tfrac{4}{243} \times 729^{5/6} \\ \Rightarrow \quad x_2^{1/3} &= 4 \\ \Rightarrow \quad x_2 &= 81 \end{aligned}$$

The solution is $x_1 = 729$, $x_2 = 81$.

Problem 6. Find all first partial derivatives of each of the following
a. $f(x) \;=\; 5x_1^3 - 3x_2^2 + 30x_1$

$$\frac{\partial f(x)}{\partial x_1} = 15x_1^2 + 30$$
$$\frac{\partial f(x)}{\partial x_2} = -6x_2$$

b. $y \;=\; 2x_1^3 x_2^{1/2}$

$$\frac{\partial y}{\partial x_1} = 6x_1^2 x_2^{1/2}$$
$$\frac{\partial y}{\partial x_2} = 2x_1^3 \left(\frac{1}{2} x_2^{-1/2}\right) = x_1^3 x_2^{-1/2}$$

c. $f(x) \;=\; 11664x_1^{1/6} x_2^{1/3} - 32x_1 - 729x_2$

$$\begin{aligned}\frac{\partial f(x)}{\partial x_1} &= \frac{11664}{6} x_1^{-5/6} x_2^{1/3} - 32 \\ &= 1944x_1^{-5/6} x_2^{1/3} - 32 \\ \frac{\partial f(x)}{\partial x_2} &= \frac{11664}{3} x_1^{1/6} x_2^{-2/3} - 729 \\ &= 3888x_1^{1/6} x_2^{-2/3} - 729\end{aligned}$$

d. $f(x) \;=\; 30x_1^{2/3}x_2^{5/6} + 6x_3^{5/6}$

$$\frac{\partial f(x)}{\partial x_1} = 20x_1^{-1/3}x_2^{5/6}$$
$$\frac{\partial f(x)}{\partial x_2} = 25x_1^{2/3}x_2^{-1/6}$$
$$\frac{\partial f(x)}{\partial x_3} = 5x_3^{-1/6}$$

e. $f(x) \;=\; 30x_1^{1/3}x_2^{3/5}x_3^{1/2} - 4x_1 - 5x_2 - x_3$

$$\frac{\partial f(x)}{\partial x_1} = 10x_1^{-2/3}x_2^{3/5}x_3^{1/2} - 4$$
$$\frac{\partial f(x)}{\partial x_2} = 18x_1^{1/3}x_2^{-2/5}x_3^{1/2} - 5$$
$$\frac{\partial f(x)}{\partial x_3} = 15x_1^{1/3}x_2^{3/5}x_3^{-1/2} - 1$$

f. $f(x) = 4{x_1}^{1/2} + 4{x_2}^{1/2} + 6{x_3}^{1/2} + 3x_1 + 4{x_1}^{1/2}{x_2}^{1/2} + 6{x_1}^{1/2}{x_3}^{1/2} + 2x_2 - 2{x_2}^{1/2}{x_3}^{1/2} + 7x_3$

$$\frac{\partial f(x)}{\partial x_1} = 2x_1^{-1/2} + 3 + 2x_1^{-1/2}x_2^{1/2} + 3x_1^{-1/2}x_3^{1/2}$$
$$\frac{\partial f(x)}{\partial x_2} = 2x_2^{-1/2} + 2x_1^{1/2}x_2^{-1/2} + 2 - x_2^{-1/2}x_3^{1/2}$$
$$\frac{\partial f(x)}{\partial x_3} = 3x_3^{-1/2} + 3x_1^{1/2}x_3^{-1/2} - x_2^{1/2}x_3^{-1/2} + 7$$