ECONOMICS 207
SPRING 2008
PROBLEM SET 12
KEY

Problem 1. Consider the following matrix and vector.

$$P = \begin{bmatrix} 1 & 1 \\ 3 & 7 \end{bmatrix}, \quad p = \begin{bmatrix} -2 \\ 2 \end{bmatrix},$$

a. Use elementary row operations to find both the inverse of P and solve the equation Px=p in one set of operations.

Augment matrix P with a 2×2 identity matrix and matrix p. That is, let

$$\tilde{A} = \begin{pmatrix} P & I_{2\times 2} & p \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 & -2 \\ 3 & 7 & 0 & 1 & 2 \end{pmatrix}$$

Then use elementary row operations on $\tilde{A}$ to create an identity matrix on the left side.
To begin with, based on matrix $\tilde{A}$, subtract the first row multiplied by 3 from the second row.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & -2 \\ 3-1\times 3 & 7-1\times 3 & 0-1\times 3 & 1 & 2-(-2)\times 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 & -2 \\ 0 & 4 & -3 & 1 & 8 \end{pmatrix} \tag{1}$$

Based on the matrix on the right side of equation (1), divide the second row by 4.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & -2 \\ 0 & 4/4 & -3/4 & 1/4 & 8/4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 & -2 \\ 0 & 1 & -3/4 & 1/4 & 2 \end{pmatrix} \tag{2}$$

Based on the matrix on the right side of equation (2), subtract the second row by the first row.

$$\begin{pmatrix} 1 & 1-1 & 1-(-3/4) & 0-1/4 & -2-2 \\ 0 & 1 & -3/4 & 1/4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 7/4 & -1/4 & -4 \\ 0 & 1 & -3/4 & 1/4 & 2 \end{pmatrix} \tag{3}$$

So $P^{-1} = \begin{pmatrix} 7/4 & -1/4 \\ -3/4 & 1/4 \end{pmatrix}$ and the solutions for $P\mathbf{x} = p$ is $\mathbf{x} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$.

Date: August 25, 2008.

b. Find the determinant of the matrix P.

$$P = \begin{bmatrix} 1 & 1 \\ 3 & 7 \end{bmatrix}, \quad p = \begin{bmatrix} -2 \\ 2 \end{bmatrix},$$

$$\det[P] = 1 \times 7 - 1 \times 3 = 4$$

c. Find the inverse of the matrix P using the cofactor/adjoint method.

The adjoint matrix of P is given by

$$\begin{aligned} \text{adj}(P) &= \begin{pmatrix} 7 & -3 \\ -1 & 1 \end{pmatrix}^T \\ &= \begin{pmatrix} 7 & -1 \\ -3 & 1 \end{pmatrix} \end{aligned}$$

Then the inverse of P is given by

$$\begin{aligned} P^{-1} &= \frac{\text{adj}(P)}{\det[P]} = \frac{1}{4} \begin{pmatrix} 7 & -1 \\ -3 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 7/4 & -1/4 \\ -3/4 & 1/4 \end{pmatrix} \end{aligned}$$

d. Solve the equation Px=p using the inverse you found in part 1c

Using the inverse of P, the solution is given by

$$\begin{aligned} \mathbf{x} &= P^{-1}p \\ &= \begin{bmatrix} 7/4 & -1/4 \\ -3/4 & 1/4 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} (7/4) \times (-2) + (-1/4) \times 2 \\ (-3/4) \times (-2) + (1/4) \times 2 \end{bmatrix} \\ &= \begin{bmatrix} -4 \\ 2 \end{bmatrix} \end{aligned}$$

e. Solve the equation Px=p using Cramer's rule.

$$P = \begin{bmatrix} 1 & 1 \\ 3 & 7 \end{bmatrix}, \quad p = \begin{bmatrix} -2 \\ 2 \end{bmatrix},$$

By Cramer's rule, the solutions is given by

$$\mathbf{x} = \frac{\begin{bmatrix} \begin{vmatrix} -2 & 1 \\ 2 & 7 \end{vmatrix} \\ \\ \begin{vmatrix} 1 & -2 \\ 3 & 2 \end{vmatrix} \end{bmatrix}}{\det[P]}$$

$$= \frac{1}{4} \begin{bmatrix} -16 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

Problem 2. Consider the following matrix and vector.

$$A = \begin{bmatrix} 1 & -3 & 2 \\ -5 & 16 & -11 \\ 3 & -6 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix}$$

a. Use elementary row operations to find both the inverse of A and solve the equation Ax=b in one set of operations.

Augment matrix A with a 3×3 identity matrix and matrix b. That is, let

$$\tilde{A} = \begin{bmatrix} A & I_{3\times 3} & b \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 & 1 & 0 & 0 & 1 \\ -5 & 16 & -11 & 0 & 1 & 0 & -5 \\ 3 & -6 & 3 & 0 & 0 & 1 & 7 \end{bmatrix}$$

Then use elementary row operations on matrix $\tilde{A}$ to create an identity matrix on the left side.

To begin with, based on matrix $\tilde{A}$, add the first row multiplied by 5 to the second row; and add the first row multiplied by -3 to the third row.

$$\begin{bmatrix} 1 & -3 & 2 & 1 & 0 & 0 & 1 \\ -5+1\times 5 & 16-3\times 5 & -11+2\times 5 & 0+1\times 5 & 1 & 0 & -5+1\times 5 \\ 3+1\times(-3) & -6+(-3)\times(-3) & 3+2\times(-3) & 0+1\times(-3) & 0 & 1 & 7+1\times(-3) \end{bmatrix} \tag{4}$$

$$= \begin{bmatrix} 1 & -3 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 5 & 1 & 0 & 0 \\ 0 & 3 & -3 & -3 & 0 & 1 & 4 \end{bmatrix} \tag{5}$$

Based on the matrix on the right side of equation (5), subtract the second row multiplied by 3 from the third row.

$$\begin{bmatrix} 1 & -3 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 5 & 1 & 0 & 0 \\ 0 & 3-1\times 3 & -3-(-1)\times 3 & -3-5\times 3 & 0-1\times 3 & 1 & 4-1\times 3 \end{bmatrix} \tag{6}$$

$$= \begin{bmatrix} 1 & -3 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 5 & 1 & 0 & 0 \\ 0 & 0 & 0 & -18 & -3 & 1 & 1 \end{bmatrix} \tag{7}$$

Since the first 3 elements of the third row of matrix on the right side of equation (7) are all zeros, to create an identity matrix is not possible. Thus, the inverse of A does not exist.

Further, the last element of the third row of matrix on the right side of equation (7) is not zero. As a result, there is no solution for the equation.

b. Find the determinant of the matrix A.

$$A = \begin{bmatrix} 1 & -3 & 2 \\ -5 & 16 & -11 \\ 3 & -6 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix}$$

$$\begin{aligned} \det[A] &= \begin{vmatrix} 16 & -11 \\ -6 & 3 \end{vmatrix} - (-3) \begin{vmatrix} -5 & -11 \\ 3 & 3 \end{vmatrix} + 2 \begin{vmatrix} -5 & 16 \\ 3 & -6 \end{vmatrix} \\ &= (48 - 66) + 3\,(-15 + 33) + 2\,(30 - 48) \\ &= 0 \end{aligned}$$

c. Find the inverse of the matrix A using the cofactor/adjoint method.

If the inverse of the matrix A does exist, it is given by

$$A^{-1} = \frac{\text{adj}(A)}{\det[A]}$$

However, $\det[A] = 0$. As a result, the inverse of the matrix A does not exist.

d. Solve the equation Ax=b using the inverse you found in part 2c

$$A = \begin{bmatrix} 1 & -3 & 2 \\ -5 & 16 & -11 \\ 3 & -6 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix}$$

Since the determinant of A is zero, there is no solution for the equation.

e. Solve the equation Ax=b using Cramer's rule.

Since the determinant of A is zero, there is no solution for the equation.

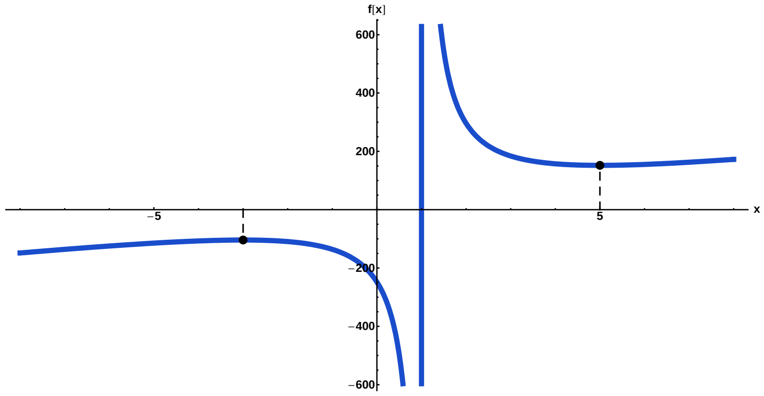
Problem 3. For each of the following problems, find the critical points. For each critical point state whether the function is at a relative maximum, relative minimum, or otherwise. Check to see if there are potential points of inflection **at points other than** critical points.

a. $f(x) \;=\; \dfrac{\left(4x^2 - 2x + 62\right)}{\frac{x}{4} - \frac{1}{4}}$.

You need not find the points of inflection for this problem. Hint: The simplified version of the first derivative is $\dfrac{\left(x^2 - 2x - 15\right)}{(\frac{x}{4} - \frac{1}{4})^2} = \dfrac{16\left(x^2 - 2x - 15\right)}{(x-1)^2}$

You should find the second derivative of f(x) but here is the answer: $\dfrac{512}{(-1+x)^3}$.

FIGURE 1. $f(x) \;=\; \dfrac{\left(4x^2 - 2x + 62\right)}{\frac{x}{4} - \frac{1}{4}}$



First, simplify $f(x) = \dfrac{\left(4x^2 - 2x + 62\right)}{\frac{x}{4} - \frac{1}{4}}$.

$$\begin{array}{r|l} & \quad\;\; 16x \quad + 8 \\ \hline \frac{1}{4}x - \frac{1}{4}) & \;\; 4x^2 - 2x + 62 \\ & - 4x^2 + 4x \\ \hline & \qquad\quad 2x + 62 \\ & \qquad - 2x \quad + 2 \\ \hline & \qquad\qquad\quad 64 \end{array}$$

That is,

$$\begin{aligned} f(x) &= 16x + 8 + 64\left(\frac{x}{4} - \frac{1}{4}\right)^{-1} \\ &= 16x + 8 + 256(x-1)^{-1} \end{aligned}$$

So the derivatives of $f(x)$ are given by

$$f'(x) = 16 - 256(x-1)^{-2} \tag{8}$$

$$f''(x) = 512(x-1)^{-3} \tag{9}$$

Set the first derivative, equation (8), to zero.

$$\begin{aligned} f'(x) = 16 - 256(x-1)^{-2} &= 0 \\ \Rightarrow \qquad (x-1)^{-2} &= 16/256 = 1/16 \\ \Rightarrow \qquad (x-1)^2 &= 16 \\ \Rightarrow \qquad (x-1) &= \pm 4 \\ \Rightarrow \qquad x = 5 \quad &\text{or} \quad x = -3 \end{aligned}$$

Check the second derivative, equation (9), for $x = 5$ and $x = -3$ respectively.

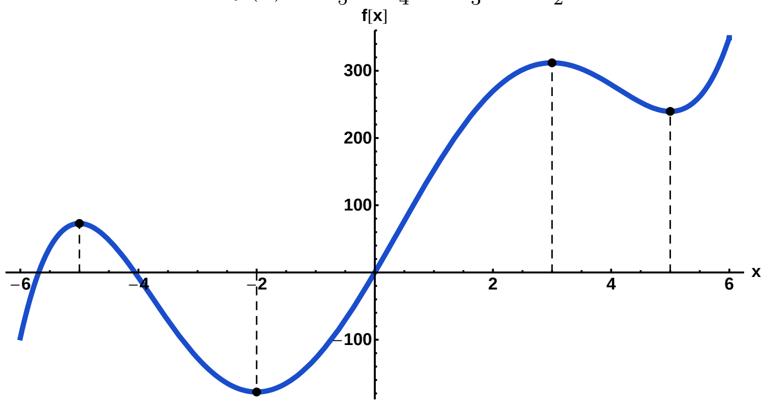
$$f''(5) = 512(5-1)^{-3} > 0$$

$$f''(-3) = 512(-3-1)^{-3} < 0$$

So $x = -3$ is a local maximum point; $x = 5$ is a local minimum point.

b. $f(x) \;=\; \frac{x^5}{5} - \frac{x^4}{4} - \frac{31x^3}{3} + \frac{25x^2}{2} + 150x$
You need not find the points of inflection for this problem as they are all imaginary numbers.
Hint: You will need to use long division twice.

FIGURE 2. $f(x) \;=\; \frac{x^5}{5} - \frac{x^4}{4} - \frac{31x^3}{3} + \frac{25x^2}{2} + 150x$



The derivatives of $f(x) = \frac{x^5}{5} - \frac{x^4}{4} - \frac{31x^3}{3} + \frac{25x^2}{2} + 150x$ are given by

$$f'(x) = x^4 - x^3 - 31x^2 + 25x + 150 \tag{10}$$

$$f''^{(x)} = 4x^3 - 3x^2 - 62x + 25 \tag{11}$$

Set the first derivative, equation (10), to be zero.

$$f'(x) = x^4 - x^3 - 31x^2 + 25x + 150 = 0 \tag{12}$$

By trying, it can be found out that $x = 3$ is a solution. So $f'(x) = x^4 - x^3 - 31x^2 + 25x + 150$ can be factorized using $(x - 3)$ as follows.

$$\begin{array}{r|l}
 & x^3 \;+ 2x^2 - 25x \;- 50 \\ \hline
x-3) & x^4 \;\; - x^3 - 31x^2 + 25x + 150 \\
 & -x^4 + 3x^3 \\ \hline
 & 2x^3 - 31x^2 \\
 & -2x^3 \;+ 6x^2 \\ \hline
 & -25x^2 + 25x \\
 & 25x^2 - 75x \\ \hline
 & -50x + 150 \\
 & 50x - 150 \\ \hline
 & 0
\end{array}$$

Still by trying, it can be found that $x^3+2x^2-25x-50$ can be factorized by $(x+2)$ as follows.

$$\begin{array}{r|l} & x^2 \qquad\quad - 25 \\ \hline x+2) & x^3+2x^2-25x-50 \\ & -x^3-2x^2 \\ \hline & \qquad\qquad -25x-50 \\ & \qquad\qquad\ \ 25x+50 \\ \hline & \qquad\qquad\qquad\quad 0 \end{array}$$

Then we can solve the equation $f'(x)=0$.

$$\begin{aligned} f'(x)=x^4-x^3-31x^2+25x+150 &= 0 \\ \Rightarrow \qquad (x-3)(x+2)(x^2-25) &= 0 \\ \Rightarrow \qquad x=3 \quad \text{or} \quad x=-2 \quad & \text{or} \quad x=\pm 5 \end{aligned}$$

Check the second derivative, equation (11), for $x=3$, $x=-2$, and $x=\pm 5$ respectively.

$$f''(3)=4\times 3^3-3\times 3^2-62\times 3+25=-80$$

$$f''(-2)=4\times(-2)^3-3\times(-2)^2-62\times(-2)+25=105$$

$$f''(5)=4\times 5^3-3\times 5^2-62\times 3+25=140$$

$$f''(-5)=4\times(-5)^3-3\times(-5)^2-62\times(-5)+25=-240$$

Therefore, $x=3$ and $x=-5$ are local maximum points; $x=-2$ and $x=5$ are local minimum points.

Problem 4. Solve the following system of equations.

$$1764x_1^{-2/3}x_2^{1/4} - 108 = 0$$
$$1323x_1^{1/3}x_2^{-3/4} - 343 = 0$$

Hint: $x_1 = 343$.

Rearrange the first equation.

$$\begin{aligned} 1764x_1^{-2/3}x_2^{1/4} - 108 &= 0 \\ \Rightarrow \quad x_1^{-2/3}x_2^{1/4} &= 108/1764 = 3/49 \\ \Rightarrow \quad x_2^{1/4} &= \tfrac{3}{49}x_1^{2/3} \\ \Rightarrow \quad x_2^{-3/4} &= \left(\tfrac{3}{49}x_1^{2/3}\right)^{-3} \end{aligned}$$

Substitute $x_2^{-3/4} = \left(\frac{3}{49}x_1^{2/3}\right)^{-3}$ into the second equation.

$$\begin{aligned} 1323x_1^{1/3}x_2^{-3/4} - 343 &= 0 \\ \Rightarrow \quad 1323x_1^{1/3}\left(\tfrac{3}{49}x_1^{2/3}\right)^{-3} - 343 &= 0 \\ \Rightarrow \quad x_1^{-5/3} &= \tfrac{343}{1323}\tfrac{3^3}{49^3} = 7^{-5} \\ \Rightarrow \quad x_1 &= 7^3 = 343 \end{aligned}$$

Substitute $x_1 = 343$ into $x_2^{1/4} = \frac{3}{49}x_1^{2/3}$.

$$\begin{aligned} x_2^{1/4} &= \tfrac{3}{49}x_1^{2/3} \\ \Rightarrow \quad x_2^{1/4} &= \tfrac{3}{49}343^{2/3} = 3 \\ \Rightarrow \quad x_2 &= 81 \end{aligned}$$

So the solution are $x_1 = 343$, $x_2 = 81$.

Problem 5. Find all first and second partial derivatives of each of the following

a. $f(x_1, x_2) \ = \ 60x_1 + 21x_2 - x_1^2 + x_1x_2 - x_2^2$

$\frac{\partial f}{\partial x_1} = 60 - 2x_1 + x_2$	$\frac{\partial f}{\partial x_2} = 21 + x_1 - 2x_2$
$\frac{\partial^2 f}{\partial x_1 \partial x_1} = -2$	$\frac{\partial^2 f}{\partial x_1 \partial x_2} = 1$
$\frac{\partial^2 f}{\partial x_2 \partial x_1} = 1$	$\frac{\partial^2 f}{\partial x_2 \partial x_2} - 2$

b. $f(x_1, x_2) \;=\; f(x_1, x_2) \;=\; 80x_1 + 22x_2 - x_1^2 + 2x_1x_2 - 2x_2^2$

$\frac{\partial f}{\partial x_1} = 80 - 2x_1 + 2x_2$	$\frac{\partial f}{\partial x_2} = 22 + 2x_1 - 4x_2$
$\frac{\partial^2 f}{\partial x_1 \partial x_1} = -2$	$\frac{\partial^2 f}{\partial x_1 \partial x_2} = 2$
$\frac{\partial^2 f}{\partial x_2 \partial x_1} = 2$	$\frac{\partial^2 f}{\partial x_2 \partial x_2} = -4$

c. $f(x_1, x_2) = 10x_1 + 9x_2 - x_1^2 + x_1x_2 - 2x_2^2$

$\frac{\partial f}{\partial x_1} = 10 - 2x_1 + x_2$	$\frac{\partial f}{\partial x_2} = 9 + x_1 - 4x_2$
$\frac{\partial^2 f}{\partial x_1 \partial x_1} = -2$	$\frac{\partial^2 f}{\partial x_1 \partial x_2} = 1$
$\frac{\partial^2 f}{\partial x_2 \partial x_1} = 1$	$\frac{\partial^2 f}{\partial x_2 \partial x_2} = -4$

d. $f(x_1, x_2, x_3) = 100x_1^{1/2}x_2^{1/5}x_3^{1/4} - 6x_1 - 5x_2 - 2x_3$

$\frac{\partial f}{\partial x_1} = 50x_1^{-1/2}x_2^{1/5}x_3^{1/4} - 6$	$\frac{\partial f}{\partial x_2} = 20x_1^{1/2}x_2^{-4/5}x_3^{1/4} - 5$	$\frac{\partial f}{\partial x_3} = 25x_1^{1/2}x_2^{1/5}x_3^{-3/4} - 2$
$\frac{\partial^2 f}{\partial x_1 \partial x_1} = -25x_1^{-3/2}x_2^{1/5}x_3^{1/4}$	$\frac{\partial^2 f}{\partial x_1 \partial x_2} = 10x_1^{-1/2}x_2^{-4/5}x_3^{1/4}$	$\frac{\partial^2 f}{\partial x_1 \partial x_3} = \frac{25}{2}x_1^{-1/2}x_2^{1/5}x_3^{-3/4}$
$\frac{\partial^2 f}{\partial x_2 \partial x_1} = 10x_1^{-1/2}x_2^{-4/5}x_3^{1/4}$	$\frac{\partial^2 f}{\partial x_2 \partial x_2} = -16x_1^{1/2}x_2^{-9/5}x_3^{1/4}$	$\frac{\partial^2 f}{\partial x_2 \partial x_3} = 5x_1^{1/2}x_2^{-4/5}x_3^{-3/4}$
$\frac{\partial^2 f}{\partial x_3 \partial x_1} = \frac{25}{2}x_1^{-1/2}x_2^{1/5}x_3^{-3/4}$	$\frac{\partial^2 f}{\partial x_3 \partial x_2} = 5x_1^{1/2}x_2^{-4/5}x_3^{-3/4}$	$\frac{\partial^2 f}{\partial x_3 \partial x_3} = -\frac{75}{4}x_1^{1/2}x_2^{1/5}x_3^{-7/4}$

Problem 6. For each of the following problems, write an equation that represents profit as a function of the two inputs x_1 and x_2. Write it in the form π = pf(x_1,x_2)-w_1x_1-w_2x_2 and then simplify the expression. Then find all first and second partial derivatives of the function at the specified point.

a.

$$\begin{aligned} f(x_1, x_2) &= 60x_1 + 21x_2 - x_1^2 + x_1x_2 - x_2^2 \\ p &= 4 \\ w_1 &= 60, \qquad w_2 = 12 \\ x_1 &= 36, \qquad x_2 = 27 \end{aligned}$$

$$\begin{aligned} \pi &= 4\left(60x_1 + 21x_2 - x_1^2 + x_1x_2 - x_2^2\right) - 60x_1 - 12x_2 \\ &= 180x_1 + 72x_2 - 4x_1^2 + 4x_1x_2 - 4x_2^2 \end{aligned}$$

$\frac{\partial \pi}{\partial x_1} = 180 - 8x_1 + 4x_2$	$\frac{\partial \pi}{\partial x_2} = 72 + 4x_1 - 8x_2$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -8$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 4$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 4$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = 8$

b.

$$
\begin{aligned}
f(x_1, x_2) &= 80x_1 + 22x_2 - x_1^2 + 2x_1x_2 - 2x_2^2 \\
p &= 4 \\
w_1 &= 24, \qquad w_2 = 16 \\
x_1 &= 83, \qquad x_2 = 46
\end{aligned}
$$

$$
\begin{aligned}
\pi &= 4(80x_1 + 22x_2 - x_1^2 + 2x_1x_2 - 2x_2^2) - 24x_1 - 16x_2 \\
&= 356x_1 + 72x_2 - 4x_1^2 + 8x_1x_2 - 8x_2^2
\end{aligned}
$$

$\frac{\partial \pi}{\partial x_1} = 356 - 8x_1 + 8x_2$	$\frac{\partial \pi}{\partial x_2} = 72 + 8x_1 - 16x_2$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -8$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 8$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 8$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -16$

c.

$$f(x_1,x_2) = 10x_1 + 9x_2 - x_1^2 + x_1x_2 - 2x_2^2$$
$$p = 5$$
$$w_1 = 30, \quad w_2 = 20$$
$$x_1 = 3, \quad x_2 = 2$$

$$\pi = 5\left(10x_1 + 9x_2 - x_1^2 + x_1x_2 - 2x_2^2\right) - 30x_1 - 20x_2$$
$$= 20x_1 + 25x_2 - 5x_1^2 + 5x_1x_2 - 10x_2^2$$

$\frac{\partial \pi}{\partial x_1} = 20 - 10x_1 + 5x_2$	$\frac{\partial \pi}{\partial x_2} = 25 + 5x_1 - 20x_2$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -10$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 5$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 5$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -20$

d.

$$f(x_1, x_2) = x_1^{1/3} x_2^{1/4}$$
$$p = 5292$$
$$w_1 = 108, \quad w_2 = 343$$
$$x_1 = 343, \quad x_2 = 81$$

$$\pi = 5292 x_1^{1/3} x_2^{1/4} - 108 x_1 - 343 x_2$$

$\frac{\partial \pi}{\partial x_1} = 1764 x_1^{-2/3} x_2^{1/4} - 108$	$\frac{\partial \pi}{\partial x_2} = 1323 x_1^{1/3} x_2^{-3/4} - 343$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -1176 x_1^{-5/3} x_2^{1/4}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 441 x_1^{-2/3} x_2^{-3/4}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 441 x_1^{-2/3} x_2^{-3/4}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -\frac{3969}{4} x_1^{1/3} x_2^{-7/4}$

In this table fill in values of x_1 and x_2 given to obtain numerical answers.

$\frac{\partial \pi}{\partial x_1} = 1764 \times 343^{-2/3} \times 81^{1/4} - 108 = 0$	$\frac{\partial \pi}{\partial x_2} = 1323 \times 343^{1/3} \times 81^{-3/4} - 343 = 0$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -1176 \times 343^{-5/3} \times 81^{1/4} = -\frac{72}{343}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 441 \times 343^{-2/3} \times 81^{-3/4} = \frac{1}{3}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = \frac{1}{3}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -\frac{3969}{4} \times 343^{1/3} \times 81^{-7/4} = -\frac{343}{108}$

e.

$$f(x_1, x_2) = x_1^{1/2} x_2^{1/4}$$
$$p = 288$$
$$w_1 = 32, \qquad w_2 = 81$$
$$x_1 = 81, \qquad x_2 = 16$$

$$\pi = 288 x_1^{1/2} x_2^{1/4} - 32 x_1 - 81 x_2$$

$\frac{\partial \pi}{\partial x_1} = 144 x_1^{-1/2} x_2^{1/4} - 32$	$\frac{\partial \pi}{\partial x_2} = 72 x_1^{1/2} x_2^{-3/4} - 81$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -72 x_1^{-3/2} x_2^{1/4}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 36 x_1^{-1/2} x_2^{-3/4}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = 36 x_1^{-1/2} x_2^{-3/4}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -54 x_1^{1/2} x_2^{-7/4}$

In this table fill in values of x_1 and x_2 given to obtain numerical answers.

$\frac{\partial \pi}{\partial x_1} = 144 \times 81^{-1/2} \times 16^{1/4} - 32 = 0$	$\frac{\partial \pi}{\partial x_2} = 72 \times 81^{1/2} \times 16^{-3/4} - 81 = 0$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} = -72 \times 81^{-3/2} \times 16^{1/4} = -\frac{16}{81}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} = 36 \times 81^{-1/2} \times 16^{-3/4} = \frac{1}{2}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} = \frac{1}{2}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} = -54 \times 81^{1/2} \times 16^{-7/4} = -\frac{243}{64}$