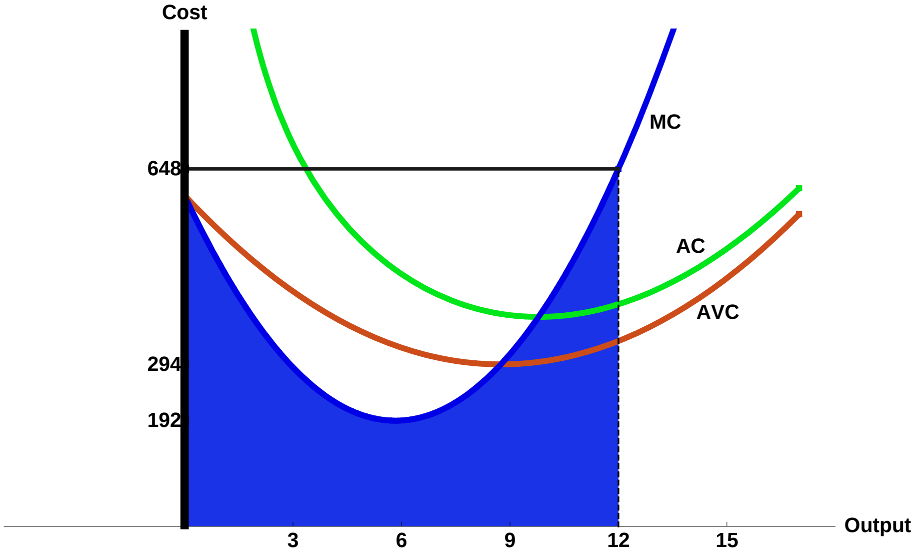
ECONOMICS 207
SPRING 2008
PROBLEM SET 13

Problem 1. The cost function for a firm is a rule or mapping that tells the minimum total cost of production of any output level produced by the firm for a fixed level of input prices. If the variable y represents the output of the firm, then the cost function is given by c(y,w) or c(y). Marginal cost represents the change in the cost of production for the firm as output changes and is given by the derivative of the cost function with respect to output, i.e., Marginal Cost(MC) $= \frac{dc(y)}{dy}$. A graph of price and marginal cost is given in figure 1.

FIGURE 1. Profit Maximization Given Marginal Cost



Consider a competitive firm with the following technology (as represented by its cost function) and output price.

$$price \;=\; p \;= \$648$$
$$cost \;=\; c(y) \;= 800 \;+\; 600y \;-\; 70y^2 \;+\; 4y^3$$

Date: April 21, 2008.

a. Write down an equation for profit for this firm and find the potential levels of output that maximize profit.

b. Using the second order conditions from profit maximization, determine which of the levels of output from part a that maximize profit?

c. Explain in words why setting price equal to marginal cost and solving for the optimal output y gives the same answers as taking the derivative of profit with respect to y, setting the result equal to zero and solving for the optimal y. Remember that

$$Profit = py - c(y)$$
$$Profit = 648y - \left[800 + 600y - 70\,y^2 + 4y^3\right]$$

d. Consider a competitive firm with the following technology (as represented by its cost function) and output price.

$$price = p = \$648$$
$$cost = c(y) = 800 + 600y - 70y^2 + 4y^3$$

Without writing down an equation for profit, find the levels of output which potentially maximize profit using what you have learned in general about profit maximization for a competitive firm.

e. Given that you have no second order conditions from profit maximization per se, make a coherent argument about which of the two potential output levels you found in part d maximizes profit.

f. Integrate the marginal cost function you found in part d to obtain the variable cost function and then use it to show that the level of variable cost for this firm when it maximizes profit with a price of \$648 is \$4032.

g. Cross-hatch the area represented by variable cost in part f and explain why the integral of the marginal cost function or the area under the marginal cost curve represents variable cost.

h. What is revenue for this profit maximizing firm?

i. What is producer surplus for this profit maximizing firm?

j. Shade this level of producer surplus in Figure 1.

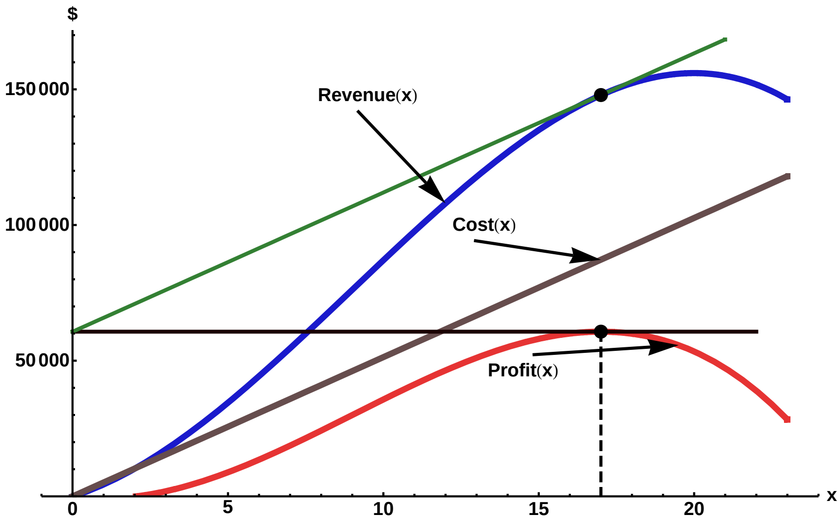
Problem 2. In the following problem you are given a production function for a firm where y is the level of output and x is the level of the variable input. You are given the price (p) of the output and the price (w) of the single variable input.

$$output\ price = p = 10$$

$$input\ price = w = 5130$$

$$y = output = f(x) = 360x + 81x^2 - 3x^3$$

FIGURE 2. Revenue, Cost and Profit



a. Find values of x that potentially maximize **output** for this firm.

b. Show which values of x in part a actually maximize output.

c. Write down an equation that represents profit for the firm.

d. Maximize this function by taking its derivative with respect to the variable input x and setting the resulting equation equal to zero.

e. If you identify more than one critical value from setting the first derivative of profit equal to zero, show which ones, if any, maximize profit.

f. Explain in words why the value of the marginal product for this firm is equal to the price of the single variable input at the profit maximizing level of input use. You can use the following information in explaining this phenomenon. Say something about the benefits of using an input not being less than the cost of the input.

$$\begin{aligned} Output =& y = f(x) \\ MP =& \text{Marginal Product} = \frac{d\,f(x)}{dx} = f'(x) = \frac{\Delta y}{\Delta x} \\ Revenue =& pf(x) \\ Cost =& wx \\ Profit = \pi =& Revenue - Cost = pf(x) - wx \\ \frac{d\pi}{dx} =& \end{aligned}$$

g. Using the information from part 2f, explain why the slope of total revenue in figure 2 is equal to the slope of total cost at the profit maximizing level of input use.

Problem 3. For each of the following problems, write an equation that represents profit as a function of the two inputs x_1 and x_2.

Write it in the form $\pi = pf(x_1,x_2)-w_1x_1-w_2x_2$ and then simplify the expression. Then find all first and second partial derivatives of the function. Then set the partial derivatives with respect to x_1 and x_2 equal to zero and solve the equations for the levels of x_1 and x_2 that maximize profit. Then show that the level you found actually maximizes profit.

a.

$$\begin{aligned} f(x_1,x_2) &= 20x_1 + 25x_2 - x_1^2 + x_1x_2 - x_2^2 \\ p &= 4 \\ w_1 &= 40, \qquad w_2 = 24 \\ x_1 &= 13, \qquad x_2 = 16, \qquad y = 443 \end{aligned}$$

$\frac{\partial \pi}{\partial x_1}$	$\frac{\partial \pi}{\partial x_2}$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2}$

Find potential profit maximizing levels of x_1 and x_2.

By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

b.

$$
\begin{aligned}
f(x_1, x_2) &= 100x_1 \;+\; 50x_2 \;-\; 2x_1^2 \;+\; x_1x_2 \;-\; 2x_2^2 \\
p &= 4 \\
w_1 &= 90, \qquad w_2 \;=\; 60 \\
x_1 &= 23, \qquad x_2 \;=\; \frac{29}{2}, \qquad y = 1880
\end{aligned}
$$

$\frac{\partial \pi}{\partial x_1}$	$\frac{\partial \pi}{\partial x_2}$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1}$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2}$

Find potential profit maximizing levels of x_1 and x_2.

By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

c.

$$
\begin{aligned}
f(x_1, x_2) &= x_1^{1/4} x_2^{1/2} \\
p &= 8 \\
w_1 &= 2, \qquad w_2 \ = \ 1 \\
x_1 &= 16, \qquad x_2 \ = \ 64, \qquad y = 16
\end{aligned}
$$

$\frac{\partial \pi}{\partial x_1} =$	$\frac{\partial \pi}{\partial x_2} =$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} =$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} =$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} =$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} =$

Find potential profit maximizing levels of x_1 and x_2.

In this table fill in values of x_1 and x_2 given to obtain numerical answers for the Hessian matrix.

$\frac{\partial \pi}{\partial x_1} =$	$\frac{\partial \pi}{\partial x_2} =$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} =$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} =$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} =$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} =$

By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

d.

$$\begin{aligned} f(x_1, x_2) &= x_1^{1/2} x_2^{1/4} \\ p &= 640 \\ w_1 &= 128, \qquad w_2 = 25 \\ x_1 &= 100, \qquad x_2 = 256, \qquad y = 40 \end{aligned}$$

$\frac{\partial \pi}{\partial x_1} =$	$\frac{\partial \pi}{\partial x_2} =$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} =$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2}$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1}$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} =$

Find potential profit maximizing levels of x_1 and x_2.

In this table fill in values of x_1 and x_2 given to obtain numerical answers for the Hessian matrix.

$\frac{\partial \pi}{\partial x_1} =$	$\frac{\partial \pi}{\partial x_2} =$
$\frac{\partial^2 \pi}{\partial x_1 \partial x_1} =$	$\frac{\partial^2 \pi}{\partial x_1 \partial x_2} =$
$\frac{\partial^2 \pi}{\partial x_2 \partial x_1} =$	$\frac{\partial^2 \pi}{\partial x_2 \partial x_2} =$

By evaluating the Hessian matrix of the profit equation at the critical values, verify the optimal levels of x_1 and x_2.

Problem 4. Find the listed partial derivatives of each of the following functions.

a. $\mathcal{L}(x_1, x_2, \lambda) \;=\; 40x_1 + 24x_2 - \lambda\left(20x_1 + 25x_2 - x_1^2 + x_1x_2 - x_2^2 - 443\right)$

$\dfrac{\partial\mathcal{L}(x_1,x_2,\lambda)}{\partial x_1}$	$\dfrac{\partial\mathcal{L}(x_1,x_2,\lambda)}{\partial x_2}$	$\dfrac{\partial\mathcal{L}(x_1,x_2,\lambda)}{\partial \lambda}$
$\frac{\partial^2\mathcal{L}}{\partial x_1 \partial x_1}$	$\frac{\partial^2\mathcal{L}}{\partial x_1 \partial x_2}$	- $\frac{\partial^2\mathcal{L}}{\partial x_1 \partial \lambda}$
$\frac{\partial^2\mathcal{L}}{\partial x_2 \partial x_1}$	$\frac{\partial^2\mathcal{L}}{\partial x_2 \partial x_2}$	- $\frac{\partial^2\mathcal{L}}{\partial x_2 \partial \lambda}$
- $\frac{\partial^2\mathcal{L}}{\partial \lambda \partial x_1}$	- $\frac{\partial^2\mathcal{L}}{\partial \lambda \partial x_2}$	- $\frac{\partial^2\mathcal{L}}{\partial \lambda \partial \lambda}$

b. $\mathcal{L}(x_1, x_2, \lambda) \;=\; \left(100x_1 \;+\; 50x_2 \;-\; 2x_1^2 \;+\; x_1x_2 \;-\; 2x_2^2\right) - \lambda\,(90x_1 + 60x_2 - 1980)$

$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1}$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2}$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda}$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} =$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} =$	$-\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} =$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1} =$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} =$	$-\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda} =$
$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1}$	$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2}$	$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda}$

c. $\mathcal{L}(x_1, x_2, \lambda) \ = \ 2x_1 + x_2 - \lambda \left(x_1^{1/4} x_2^{1/2} - 16 \right)$

$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1}$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2}$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda}$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} =$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} =$	$-\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} =$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} =$	$-\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda}$
$-\ \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1}$	$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2}$	$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda}$

d. $\mathcal{L}(x_1, x_2, \lambda) \;=\; \left(x_1^{1/3} x_2^{1/4}\right) - \lambda\,(128x_1 + 25x_2 - 19200)$

$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_1}$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial x_2}$	$\dfrac{\partial \mathcal{L}(x_1, x_2, \lambda)}{\partial \lambda}$
$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_1} =$	$\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial x_2} =$	$-\frac{\partial^2 \mathcal{L}}{\partial x_1 \partial \lambda} =$
$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_1}$	$\frac{\partial^2 \mathcal{L}}{\partial x_2 \partial x_2} =$	$-\ \frac{\partial^2 \mathcal{L}}{\partial x_2 \partial \lambda}$
$-\ \frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_1}$	$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial x_2}$	$-\frac{\partial^2 \mathcal{L}}{\partial \lambda \partial \lambda}$

Problem 5. Consider the following matrix and vector.

$$V = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 6 \\ -2 & -3 & -3 \end{bmatrix},$$

$$v = \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix},$$

a. Find the determinant of the matrix V.

b. Find the inverse of the matrix V using the adjoint method.

c. Using the inverse from part b, solve the system of equations

$$V \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = v$$

$$V = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 6 \\ -2 & -3 & -3 \end{bmatrix}, \quad v = \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}$$

d. Using Cramer's rule, solve the system of equations

$$V \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = v$$

$$V = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 4 & 6 \\ -2 & -3 & -3 \end{bmatrix}, \quad v = \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}$$